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Schrödinger operators and de Branges spaces

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Abstract

We present an approach to de Branges's theory of Hilbert spaces of entire functions that emphasizes the connections to the spectral theory of differential operators. The theory is used to discuss the spectral representation of one-dimensional Schrödinger operators and to solve the inverse spectral problem.

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1. Introduction

In this paper, I will discuss the general direct and inverse spectral theory of one-dimensional Schrödinger operators $H = -d^2/dx^2 + V(x)$ from the point of view of de Branges's theory of Hilbert spaces of entire functions. In particular, I will present a new solution of the inverse spectral problem. Basically, we will obtain a local version of the Gelfand–Levitan characterization [14] of the spectral data of one-dimensional Schrödinger operators. However, our treatment is quite different from that of Gelfand–Levitan.

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I have tried to pursue two goals in this paper. First of all, I will emphasize the connections between de Branges's theory of Hilbert spaces of entire functions and the spectral theory of differential operators from the very beginning, and I hope that this leads to a concrete and accessible introduction to de Branges's results, at least for people with a background similar to mine. My treatment of de Branges's theory is, of course, by no means intended to be a replacement for the deeper and more general, but also more abstract and demanding treatment of de Branges himself in [5–8] and especially [9].

The second and perhaps more important goal is to give a new view on the (especially inverse) spectral theory of one-dimensional Schrödinger operators by recognizing it as a part of a larger picture. More specifically, I believe that one of de Branges's major results (namely, Theorem 7.3) may be interpreted as the mother of many inverse theorems. In this paper, we will use it to discuss the inverse theory for Schrödinger operators, but I think one can discuss along these lines the inverse theory of other operators as well, provided there is a good characterization of the spectral data that occur. In particular, it should be possible to give such a treatment for the one-dimensional Dirac operator.

The treatment of the inverse spectral problem given in this paper is neither short nor elementary, the major thrust really is the new picture it provides. It is not short because there are computational parts and technical issues (mainly in Sections 13–15) that need to be taken care of. However, I think that the general strategy, which will be explained in Section 9, is quite transparent. Our treatment is not elementary, either, because it depends on the machinery of de Branges spaces and at least two major results from this theory (Theorems 7.3 and 7.4), which will not be proved here.

To place this paper into context, let me mention some work on related topics. De Branges's results from [5–9] are rather complete, so not much has been added since as far as the general, abstract theory is concerned. Dym and Dym-McKean [11,12] also use de Branges spaces to study certain differential operators, and they give independent introductions de Branges's results. The theory of de Branges spaces is intimately connected with the theory of the so-called canonical systems (also known as Zakharov–Shabat systems), and there exists a considerable literature on this subject. See, for instance, [18,26] and the references cited therein. Sakhnovich's book [26] in fact discusses more general systems, and a study of these systems in the spirit of de Branges spaces is carried out in [1]. As for the inverse spectral theory of one-dimensional Schrödinger operators, there is the classical work of Gelfand–Levitan mentioned above [14]. Important improvements are due to Levitan and Gasymov [22], and further developments of this line of attack may be found in [28,29]. For modern expositions of the Gelfand–Levitan theory, we refer the reader to Chapter 2 of either [21] or [23]. A different approach—which so far has been used to attack uniqueness questions, but in principle also gives a procedure for reconstructing the potential from the spectral data—was recently developed by Simon, partly in collaboration with Gesztesy [15,27]. This approach emphasizes the role of large z asymptotics and is quite different from both [14] and the approach used here. However, we will see some connections in Section 4. Actually, after the preparation of the first version of this paper, it turned out that these connections have consequences concerning an open question from [27]—see [25] for more on this. Finally, for still another recent treatment of uniqueness questions, see [20].

This paper is organized as follows. We define de Branges spaces and establish some basic properties in the following section. In Section 3, we then discuss classical material on the spectral representation of Schrödinger operators from this point of view. This gives an immediate intuitive understanding of de Branges spaces, and it also provides an aesthetically pleasing picture of the spectral representation. Moreover, this material is then used to derive conditions on the spectral data (which are related to the Gelfand–Levitan conditions). The local approach suggested by the theory of de Branges spaces simplifies this treatment considerably. Here, by "local" we roughly mean that instead of studying the problem on the half line $(0, \infty)$ at one stroke, we study the problems on $(0, N)$ for arbitrary $N > 0$.

In Section 5, we state the inverse spectral theorem, which is the converse of the results of Section 4. According to the general philosophy of this paper, this inverse spectral theorem will also be formulated in the language of de Branges spaces. The proof requires preparatory material; this is presented in Sections 6–8. In particular, in Section 7 we state, without proof, four theorems on de Branges spaces on which our treatment of the inverse problem will crucially depend. In Section 9, we start the proof of the inverse spectral theorem, and we explain the general strategy. This proof is then carried out in Sections 11–16. In Section 10, we prepare for the proof by a discussion of canonical systems in the style of the treatment of Section 3. In Section 17, we discuss the implications of our results for the spectral measures of Schrödinger operators on the half line $(0, \infty)$. We do this mainly in order to clarify the relations to the Gelfand–Levitan theory. Section 18 contains some remarks of a more general character. The final Section 19 presents the analogs of our results for Dirichlet boundary conditions at the origin (in the main body of the paper, we exclusively deal with Neumann boundary conditions). Dirichlet boundary conditions are important in a variety of situations; in fact, I will need this material in [25]. In Section 19, we also give a characterization of half line spectral functions for locally integrable potentials, which, in this generality, could be new.

2. Elementary properties of de Branges spaces

One way to understand de Branges spaces is to interpret them as weighted versions of Paley–Wiener spaces. This point of view is put forward in the introduction of [9]. So let us recall the Paley–Wiener Theorem. Fix $a > 0$, and define PW_a as the space of Fourier transforms $\hat{f}$ of functions $f \in L_2(-a,a)$ (where $\hat{f}(k) = (2\pi)^{-1/2} \int f(x)e^{-ikx}\, dx$). For $f \in L_2(-a,a)$, the Fourier transform $\hat{f}$, originally defined as an element of $L_2(\mathbb{R})$, uniquely extends to an entire function. The Paley–Wiener Theorem says that

$$PW_a = \Big\{F : \mathbb{C} \to \mathbb{C} : F \text{ entire}, \int_{\mathbb{R}} |F(\lambda)|^2\, d\lambda < \infty, |F(z)| \leqslant C_F e^{a|z|}\Big\}. \tag{2.1}$$

An entire function $E : \mathbb{C} \to \mathbb{C}$ is called a *de Branges function* if $|E(z)| > |E(\bar{z})|$ for all $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Note that such an E is root-free on $\mathbb{C}^+$. Now the *de Branges space* $B(E)$ based on E is defined in analogy to (2.1): It consists of the entire functions F which are square integrable on the real line with respect to the weight function $|E|^{-2}$,

$$\int_{\mathbb{R}} \left|\frac{F(\lambda)}{E(\lambda)}\right|^2 d\lambda < \infty, \tag{2.2}$$

and satisfy a growth condition at infinity. In the presence of (2.2), there are a number of ways to state this condition. To formulate this result, we need some notions from the theory of Hardy spaces. However, this subject will not play an important role in what follows. A good reference for further information on this topic is [13].

We write N_0 for the set of those functions from the Nevanlinna class N for which the point mass at infinity in the canonical factorization is non-negative. A more direct, equivalent characterization goes as follows: $f \in N$ precisely if f is holomorphic on $\mathbb{C}^+$ and can be written as the quotient of two bounded holomorphic functions on $\mathbb{C}^+$: $f = F_1/F_2$. Such an f is in N_0 if in this representation, F_2 can be chosen so that

$$\lim_{y \to \infty} \frac{\ln |F_2(iy)|}{y} = 0.$$

We will also need the Hardy space H_2 (on the upper half plane), which may be defined as follows: $f \in H_2$ precisely if f is holomorphic on $\mathbb{C}^+$ and

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx < \infty.$$

Equivalently, H_2 is the space of Fourier transforms of functions from $L_2(-\infty, 0)$.

Proposition 2.1. *Suppose that F is entire and* (2.2) *holds. Then the following are equivalent*:

(a) $|F(z)/E(z)|, |F^{\#}(z)/E(z)| \leqslant C_F(\operatorname{Im} z)^{-1/2}$ *for all* $z \in \mathbb{C}^+$.
(b) $F/E, F^{\#}/E \in N_0$.
(c) $F/E, F^{\#}/E \in H_2$.

Here, we use the notation $F^{\#}(z) = \overline{F(\bar{z})}$. By definition, an entire function F is in $B(E)$ precisely if, in addition to (2.2), one (and hence all) of these conditions holds.

In [9], de Branges uses condition (b) to define $B(E)$ (functions from N are called functions of bounded type in [9]). Condition (a) is used in [12], while (c) gives the most elegant description of $B(E)$ as

$$B(E) = \{F : \mathbb{C} \to \mathbb{C} : F \text{ entire}, \ F/E, F^{\#}/E \in H_2\}. \tag{2.3}$$

Clearly, (2.2) now follows automatically.

Proof. As $H_2 \subset N_0$, (c) implies (b). Condition (c) also implies (a) because H_2 functions admit a Cauchy type representation [13, Chapter II]:

$$\frac{F(z)}{E(z)} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} \quad (z \in \mathbb{C}^+),$$

and similarly for $F^{\#}/E$. Taking (2.2) into account, we now get (a) by applying the Cauchy–Schwarz inequality.

Now assume that (a) holds. A standard application of the residue theorem (see [12, Section 6.1] for the details) shows that

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} F(z)/E(z), & z \in \mathbb{C}^+, \\ 0, & z \in \mathbb{C}^-. \end{cases} \tag{2.4}$$

It is well known that (2.4) together with (2.2) implies that $F/E \in H_2$ [13, Exercise II.2a]. Of course, an analogous argument works for $F^{\#}/E$, so (c) holds.

Finally, we show that (b) implies (c). The canonical factorization (see again [13]) of $F/E \in N_0$ reads

$$F(z)/E(z) = e^{i\alpha} e^{ihz} B(z) g(z) S_1(z)/S_2(z), \tag{2.5}$$

where $\alpha \in \mathbb{R}$, $h \geqslant 0$, B is a Blaschke product, g is an outer function, and S_1, S_2 are the singular factors. Now F/E is meromorphic, and (2.2) prevents poles on the real line, so F/E is actually holomorphic not only on the upper half plane, but on a neighborhood of the closure of $\mathbb{C}^+$. As a consequence, $S_1 = S_2 \equiv 1$. To see this, just recall how the singular factors were constructed [13, Section II.5]. Given this, (2.2) and (2.5) together with Jensen's inequality now imply that $F/E \in H_2$ (cf. [13, Section II.5]). By the same argument, $F^{\#}/E \in H_2$. $\square$

Theorem 2.2. $B(E)$, *endowed with the inner product*

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \overline{F(\lambda)} G(\lambda) \frac{d\lambda}{|E(\lambda)|^2},$$

is a Hilbert space. Moreover, for any $z \in \mathbb{C}$, point evaluation is a bounded linear functional. More explicitly, the entire function J_z given by

$$J_z(\zeta) = \frac{\overline{E(z)}E(\zeta) - E(\bar{z})\overline{E(\bar{\zeta})}}{2i(\bar{z} - \zeta)}$$

belongs to $B(E)$ for every $z \in \mathbb{C}$, and $[J_z, F] = F(z)$ for all $F \in B(E)$.

Proof. $B(E)$ is obviously a linear space, and $[\cdot,\cdot]$ is a scalar product on $B(E)$. Also, using condition (a) from Proposition 2.1, it is not hard to see that $J_z \in B(E)$ for every $z \in \mathbb{C}$.

Now fix $F \in B(E)$. Then, as noted above, F/E obeys the Cauchy type formula (2.4). A similar computation shows that

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(\lambda)}{E^{\#}(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} 0, & z \in \mathbb{C}^+, \\ -F(z)/E^{\#}(z), & z \in \mathbb{C}^-. \end{cases}$$

Combining these equations, we see that indeed

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \overline{J_z(\lambda)} F(\lambda) \frac{d\lambda}{|E(\lambda)|^2}, \tag{2.6}$$

at least if $z \notin \mathbb{R}$. But the right-hand side of (2.6) is an entire function of z, so (2.6) must hold for all $z \in \mathbb{C}$.

It remains to prove completeness of $B(E)$. Since entire functions are already determined by their restrictions to $\mathbb{R}$, the space $B(E)$ may be viewed as a subspace of $L_2(\mathbb{R}, \pi^{-1}|E(\lambda)|^{-2}d\lambda)$. So we only need to show that $B(E)$ is closed in this larger space. To this end, observe that

$$\|J_z\|^2 = J_z(z) = \frac{|E(z)|^2 - |E(\bar{z})|^2}{4 \operatorname{Im} z}$$

remains bounded if z varies over a compact set. So if $F_n \in B(E)$ converges in norm to some $F \in L_2(\mathbb{R}, \pi^{-1}|E(\lambda)|^{-2}d\lambda)$, then $F_n(z) = \langle J_z, F_n \rangle_{L_2}$ converges uniformly on compact sets to $\langle J_z, F \rangle$, and thus $F(z) = \langle J_z, F \rangle$ defines an entire extension of $F \in L_2(\mathbb{R}, \pi^{-1}|E(\lambda)|^{-2}d\lambda)$. We can now use (2.3) and completeness of H_2 to see that F belongs to $B(E)$. $\square$

$E_a(z) = e^{-iaz}$ is a de Branges function. With this choice, we recover the Paley–Wiener space from (2.1): $PW_a = B(E_a)$. The general de Branges space $B(E)$ shares many properties with this simple example, as the full blown theory from [9] shows: $B(E)$ *always* consists of transforms of L_2 functions with bounded support. However, in the general case, one has to use eigenfunctions of a differential operator instead of the exponentials e^{ikx} and spectral measures instead of Lebesgue measure. These (rather vague) remarks will be made more precise later. Note also that the reproducing kernel J_z for $B(E_a) = PW_a$ is the Dirichlet kernel,

$$J_z(\zeta) = D_a(\bar{z} - \zeta) = \frac{\sin a(\bar{z} - \zeta)}{\bar{z} - \zeta},$$

as a brief computation shows. This is easy to understand: for general L_2 functions, convolution with D_a projects onto the frequencies in $(-a, a)$, but for functions in PW_a, these are the only frequencies that occur, so D_a acts as a reproducing kernel on this space.

There is another simple choice for E. Every polynomial without zeros in $\mathbb{C}^+ \cup \mathbb{R}$ is a de Branges function. It is clear that in this case $B(E)$ contains precisely the polynomials whose degree is smaller than that of E. Basically, the theory of these (finite dimensional) de Branges spaces is the theory of orthogonal polynomials. Many results from [9] can be viewed as generalizations of results about orthogonal polynomials.

3. Spectral representation of 1D Schrödinger operators

In this section, we show that the spaces used in the usual spectral representation of Schrödinger operators on bounded intervals are de Branges spaces. So consider the equation

$$-y''(x) + V(x)y(x) = zy(x), \tag{3.1}$$

with $V \in L_1(0, N)$. We will also be interested in the associated self-adjoint operators on $L_2(0, N)$. Throughout this paper, but with the exception of the final section, we will use Neumann boundary conditions at $x = 0$. Thus we consider the operators $H_N^\beta = -d^2/dx^2 + V(x)$ on $L_2(0, N)$ with boundary conditions

$$y'(0) = 0, \quad y(N)\sin\beta + y'(N)\cos\beta = 0.$$

We start by recalling some basic facts about the spectral representation of H_N^β. General references for this material are [4,30].

The spectrum of H_N^β is simple and purely discrete. Let $u(x, z)$ be the solution of (3.1) with the initial values $u(0, z) = 1$, $u'(0, z) = 0$ (so u satisfies the boundary condition at $x = 0$). Define the Borel measure ρ_N^β by

$$\rho_N^\beta = \sum_{\frac{u'}{u}(N,E) = -\tan\beta} \frac{\delta_E}{\|u(\cdot, E)\|_{L_2(0,N)}^2}. \tag{3.2}$$

Here, δ_E denotes the Dirac measure (i.e. $\delta_E(\{E\}) = 1, \delta_E(\mathbb{R}\setminus\{E\}) = 0$), and the sum ranges over all eigenvalues of H_N^β, and of course this interpretation also makes sense if $\beta = \pi/2$.

The operator $U : L_2(0, N) \to L_2(\mathbb{R}, d\rho_N^\beta)$, defined by

$$(Uf)(\lambda) = \int u(x, \lambda) f(x)\, dx, \tag{3.3}$$

is unitary, and $UH_N^\beta U^*$ is multiplication by λ in $L_2(\mathbb{R}, d\rho_N^\beta)$. It is a simple but noteworthy fact that the action of U depends neither on N nor on the boundary condition β. The adjoint (or inverse) of U acts as

$$(U^*F)(x) = \int_{\mathbb{R}} u(x, \lambda) F(\lambda)\, d\rho_N^\beta(\lambda), \tag{3.4}$$

for $F \in L_2(\mathbb{R}, d\rho_N^\beta)$ with finite support.

Similar statements hold for half line problems (if a potential $V \in L_{1,\mathrm{loc}}([0,\infty))$ is given, except that the construction of the spectral measure ρ is slightly more complicated. One can use, for instance, the limiting procedure of Weyl (see [4, Chapter 9]). Also, there is the distinction between the limit point and limit circle cases. In the latter case, one needs a boundary condition at infinity to get self-adjoint operators (see again [4] or [30]). In either case, U, defined by (3.3) for compactly supported $f \in L_2(0,\infty)$, extends uniquely to a unitary map $U : L_2(0,\infty) \to L_2(\mathbb{R}, d\rho)$, and we still have that UHU^* is multiplication by the variable in $L_2(\mathbb{R}, d\rho)$ (in the limit circle case, ρ and H depend on the boundary condition at infinity). Finally, for compactly supported $F \in L_2(\mathbb{R}, d\rho)$, we also still have (3.4), with ρ_N^β replaced by ρ, of course. In this paper, half line problems will sometimes be lurking in the background, but we will mainly work with problems on bounded intervals.

We now identify $L_2(\mathbb{R}, d\rho_N^\beta)$ as a de Branges space. Let

$$E_N(z) = u(N, z) + iu'(N, z).$$

Then, since $u(N, \bar{z}) = \overline{u(N, z)}$ and similarly for u',

$$\frac{\overline{E_N(z)}E_N(\zeta) - E_N(\bar{z})\overline{E_N(\bar{\zeta})}}{2i(\bar{z} - \zeta)} = \frac{\overline{u(N,z)}u'(N,\zeta) - \overline{u'(N,z)}u(N,\zeta)}{\bar{z} - \zeta}. \tag{3.5}$$

Denote the left-hand side of (3.1) by τy. We have Green's identity

$$\int_0^N (\overline{(\tau f)}g - \bar{f}\tau g) = (\overline{f(x)}g'(x) - \overline{f'(x)}g(x))|_{x=0}^{x=N},$$

and this allows us to write (3.5) in the form

$$\frac{\overline{E_N(z)}E_N(\zeta) - E_N(\bar{z})\overline{E_N(\bar{\zeta})}}{2i(\bar{z} - \zeta)} = \int_0^N \overline{u(x,z)}u(x,\zeta)\, dx.$$

Taking $z = \zeta \in \mathbb{C}^+$ shows that E_N is a de Branges function. The de Branges space based on E_N will be denoted by $S_N \equiv B(E_N)$ (S for Schrödinger). By Theorem 2.2 and the above calculation, the reproducing kernel J_z of S_N is given by

$$J_z(\zeta) = \int_0^N \overline{u(x,z)}u(x,\zeta)\,dx. \tag{3.6}$$

Theorem 3.1. *For any boundary condition β at $x = N$, the Hilbert spaces S_N and $L_2(\mathbb{R}, d\rho_N^\beta)$ are identical. More precisely, if $F(z) \in S_N$, then the restriction of F to $\mathbb{R}$ belongs to $L_2(\mathbb{R}, d\rho_N^\beta)$, and $F \mapsto F|_{\mathbb{R}}$ is a unitary map from S_N onto $L_2(\mathbb{R}, d\rho_N^\beta)$.*

Proof. Basically, the theorem is true because J_z, as given in (3.6), is the reproducing kernel for both spaces. The formal proof proceeds as follows.

Fix $\beta \in [0, \pi)$. We will usually drop the reference to this parameter (and also to N) in the notation in this proof. Let $\{\lambda_n\}$ be the eigenvalues of H_N^β; note that $\{\lambda_n\}$ supports the spectral measure $\rho = \rho_N^\beta$. We first claim that $J_z \in L_2(\mathbb{R}, d\rho)$ for every $z \in \mathbb{C}$. More precisely, by this we mean that the restriction of J_z to $\mathbb{R}$ (or $\{\lambda_n\}$) belongs to $L_2(\mathbb{R}, d\rho)$. Indeed, using (3.2) and (3.6), we obtain

$$\begin{aligned}
\|J_z\|^2_{L_2(\mathbb{R},d\rho)} &= \sum_n |J_z(\lambda_n)|^2 \rho(\{\lambda_n\}) \\
&= \sum_n |\langle u(\cdot,z), u(\cdot,\lambda_n)\rangle_{L_2(0,N)}|^2 \|u(\cdot,\lambda_n)\|^{-2}_{L_2(0,N)} \\
&= \|u(\cdot,z)\|^2_{L_2(0,N)}.
\end{aligned}$$

The last equality is Parseval's formula, which applies because the normed eigenfunctions $u(\cdot,\lambda_n)/\|u(\cdot,\lambda_n)\|$ form an orthonormal basis of $L_2(0,N)$. A similar computation shows that

$$\langle J_w, J_z\rangle_{L_2(\mathbb{R},d\rho)} = \langle u(\cdot,z), u(\cdot,w)\rangle_{L_2(0,N)} = J_z(w) = [J_w, J_z]_{S_N}.$$

By extending linearly, we thus get an isometric restriction map $V_0 : L(\{J_z : z \in \mathbb{C}\}) \to L_2(\mathbb{R}, d\rho)$, $V_0 J_z = J_z|_{\mathbb{R}}$. V_0 extends uniquely to an isometry $V : \overline{L(\{J_z : z \in \mathbb{C}\})} \to L_2(\mathbb{R}, d\rho)$. Now the finite linear combinations of the J_z are dense both in $L_2(\mathbb{R}, d\rho)$ and in S_N. In fact, as $J_{\lambda_m}(\lambda_n) = \|u(\cdot,\lambda_n)\|^2 \delta_{mn}$, the J_z already span $L_2(\mathbb{R}, d\rho)$ if z runs through the eigenvalues λ_n. As for S_N, we just note that since $[J_z, F] = F(z)$, an $F \in S_N$ that is orthogonal to all J_z's must vanish identically.

It follows that V maps S_N unitarily onto $L_2(\mathbb{R}, d\rho)$. Finally, if $F \in S_N$, then

$$(VF)(\lambda_n) = \langle V_0 J_{\lambda_n}, VF\rangle_{L_2(\mathbb{R},d\rho)} = [J_{\lambda_n}, F] = F(\lambda_n),$$

so V (originally defined by a limiting procedure) indeed just is the restriction map on the whole space. □

Recall that U from (3.3) maps $L_2(0,N)$ unitarily onto $L_2(\mathbb{R}, d\rho_N^\beta)$. Hence, by using the identification $L_2(\mathbb{R}, d\rho_N^\beta) \equiv S_N$ obtained in Theorem 3.1, we get an induced unitary map (which we still denote by U) from $L_2(0,N)$ onto S_N. We claim that this map is still given by (3.3); more precisely, for $f \in L_2(0,N)$,

$$(Uf)(z) = \int u(x,z) f(x)\, dx \quad (z \in \mathbb{C}). \tag{3.7}$$

To see this, note that (3.7) is correct for $f = u(\cdot, \lambda_n)$, where λ_n is an eigenvalue of H_N^{β}. Indeed, $u(\cdot, \lambda_n)$ is real valued, so in this case the right-hand side of (3.7) equals $J_{\lambda_n}(z)$, which clearly is in S_N. It is of course automatic that Uf, computed with formula (3.7), restricts to the right function on $\{\lambda_n\}$. Now (3.7) follows in full generality by a standard approximation argument.

As a consequence, we have the following alternate description of S_N as a set, in addition to definition (2.3):

$$S_N = \left\{ F(z) = \int_0^N u(x,z) f(x)\, dx : f \in L_2(0,N) \right\}. \tag{3.8}$$

This may be interpreted as a statement of Paley–Wiener type. Originally, S_N was defined as a space of entire functions which are square integrable on the real line with respect to a weight function and satisfy a growth condition; now (3.8) says that these function precisely arise by transforming L_2 functions with support in $(0,N)$, using the eigenfunctions $u(\cdot, z)$.

In the case of zero potential, one basically recovers the original Paley–Wiener Theorem. A still much more general result along these lines (namely, Theorem 7.3) will be discussed later.

The material developed so far has some consequences. We continue to denote the de Branges space associated with a Schrödinger equation on an interval $(0,N)$ by S_N.

Theorem 3.2. (a) *Suppose that $0 < N \leqslant N'$. Then S_N is isometrically contained in $S_{N'}$, that is, $S_N \subset S_{N'}$ and $\|F\|_{S_N} = \|F\|_{S_{N'}}$ for all $F \in S_N$.*

(b) *For any boundary condition $\beta \in [0,\pi)$ and any spectral measure ρ for the half line problem (if V is originally defined only on $(0,N)$, one may in fact also choose an arbitrary locally integrable continuation of V to $[0,\infty)$ and possibly also a boundary condition at infinity), we have that*

$$\begin{aligned} \|F\|_{S_N}^2 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|F(\lambda)|^2}{u^2(N,\lambda) + u'^2(N,\lambda)}\, d\lambda \\ &= \int_{-\infty}^{\infty} |F(\lambda)|^2\, d\rho_N^{\beta}(\lambda) = \int_{-\infty}^{\infty} |F(\lambda)|^2\, d\rho(\lambda) \end{aligned}$$

for all $F \in S_N$.

Remark. (1) Of course, the two parts of the theorem can be combined, and thus in the second part, N on the right-hand sides can be replaced by any $N' \geqslant N$.

(2) Part (b) says that S_N is embedded in $L_2(\mathbb{R}, d\mu)$ for many measures μ. In fact, one can give a description of *all* such measures μ. This description is an

analog of the Nevanlinna parametrization of the solutions of a moment problem. See [9, Theorem 30] for a very general version of this statement.

(3) It is a well known (and often useful) fact that in the limit point case, the (unique) spectral measure ρ of the half line problem can be obtained as

$$d\rho(\lambda) = \frac{1}{\pi} \lim_{N\to\infty} \frac{d\lambda}{u^2(N,\lambda) + u'^2(N,\lambda)}. \tag{3.9}$$

More precisely, (3.9) holds when integrated against continuous functions with compact support. Theorem 3.2 shows that this convergence takes place in a rather peculiar way.

(4) The fact that $\|F\|_{S_N}$ can be computed by integrating against the discrete measures ρ_N^β may be interpreted as a sampling theorem. In this context, recall that (3.8) indeed says that, in a certain sense, functions from S_N have limited bandwidth.

(5) Here, we use the term "spectral measure" in the sense of Weyl theory. Later, in Section 17, we will *define* spectral measures as those measures that satisfy the conclusion of Theorem 3.2(b).

Proof. (a) Obviously, $L_2(0,N)$ is a subspace of $L_2(0,N') = L_2(0,N) \oplus L_2(N,N')$, and the map U from (3.7) maps these spaces unitarily onto S_N and $S_{N'}$, respectively. (Here, we make essential use of the fact that the action of U is independent of N.)

(b) The first integral is the definition of the norm on S_N. The second formula gives the correct result because $S_N \equiv L_2(\mathbb{R}, d\rho_N^\beta)$ by Theorem 3.1. Finally, since $L_2(0,N)$ is isometrically contained in $L_2(0,\infty)$, the argument from the first part of this proof also shows that S_N is isometrically contained in $L_2(\mathbb{R}, d\rho)$. $\square$

4. The spaces S_N

We now analyze in more detail the de Branges spaces S_N that come from a Schrödinger equation, as discussed in the previous section. We will prove that S_N as a vector space is independent of the potential V. Moreover, the norm on S_N always is a small distortion of the norm for zero potential. In particular, the topology of S_N is also independent of V.

Along the way, we introduce the function ϕ which should be thought of as the spectral data of the Schrödinger equation. This function also plays a central role in the Gelfand–Levitan treatment of the inverse problem. The use of ϕ instead of the spectral measure (say) has many advantages. For instance, ϕ allows us to treat the problem locally: ϕ on an interval $[0,2N]$ determines and is determined by V on $[0,N]$. This is also implicit in the Gelfand–Levitan theory, although this aspect is usually not emphasized. The results of this section are basic to our approach to the inverse problem.

We assume that a potential $V \in L_1(0, N)$ is given. Then we can make the following statements about the structure of the associated space S_N.

Theorem 4.1. *As a set, S_N is given by*

$$S_N = \left\{ F(z) = \int_0^N f(t) \cos \sqrt{z}t \, dt : f \in L_2(0, N) \right\}.$$

Note that if $V \equiv 0$, then $u(x, z) = \cos \sqrt{z}x$, so the set from Theorem 4.1 is just description (3.8) of the de Branges space $S_N^{(0)}$ for zero potential. The function $f \in L_2(0, N)$ is of course uniquely determined by the corresponding $F \in S_N$.

There are also strong restrictions on the possible scalar products on the de Branges spaces coming from Schrödinger equations. This is the content of the following theorem.

We need some notation. Given a continuous, even function $\phi : [-2N, 2N] \to \mathbb{R}$, we define an integral operator $\mathcal{K}_\phi$ on $L_2(0, N)$ by

$$(\mathcal{K}_\phi f)(s) = \int_0^N K(s, t) f(t) \, dt, \tag{4.1a}$$

$$K(s, t) = \frac{1}{2}(\phi(s - t) + \phi(s + t)). \tag{4.1b}$$

$\mathcal{K}_\phi$ is self-adjoint and compact (in fact, Hilbert–Schmidt).

Theorem 4.2. *There exists a function $\phi : [-2N, 2N] \to \mathbb{R}$ which is absolutely continuous, even, and satisfies $\phi(0) = 0$, such that for all $F \in S_N$,*

$$\|F\|_{S_N}^2 = \langle f, (1 + \mathcal{K}_\phi) f \rangle_{L_2(0,N)}.$$

Here, f is related to F as in Theorem 4.1.

The requirement that ϕ be absolutely continuous on $[-2N, 2N]$ means that there exists a function $\phi' \in L_1(-2N, 2N)$ such that $\phi(x) = \phi(0) + \int_0^x \phi'(t) \, dt$ for all $x \in [-2N, 2N]$.

Later (Theorem 8.1), we will prove that Theorems 4.1 and 4.2 have a converse: the conditions formulated in these two theorems actually characterize the de Branges spaces that come from a Schrödinger equation among all de Branges spaces.

Both theorems depend on the fact that the asymptotics of the solutions of (3.1) as $|z| \to \infty$ are in leading order independent of V.

Proof of Theorem 4.1. We first show that S_N is contained in the set on the right-hand side. So let $F \in S_N$; by (3.8), F is of the form

$$F(z) = \int_0^N u(x, z) g(x) \, dx \quad (g \in L_2(0, N)). \tag{4.2}$$

By a standard asymptotic expansion, we have that

$$|u(x,z) - \cos\sqrt{z}x| \\ \leqslant |z|^{-1/2}\exp(\|V\|_{L_1(0,N)})\exp(|\mathrm{Im}\, z^{1/2}|x) \quad (0\leqslant x\leqslant N). \tag{4.3}$$

Compare, for instance, [3,24] (actually, both references do not present the result exactly in the form quoted above, but minor modifications yield (4.3)).

Define $G(k) = F(k^2)$. Then G is entire, even, and (4.2) and (4.3) imply that $|G(k)| \leqslant Ce^{N|k|}$. Moreover, again by (4.2) and (4.3),

$$G(k) = \int_0^N g(x)\cos kx\, dx + O(k^{-1})$$

for $k\in\mathbb{R}$, $k\to\infty$, so $G\in L_2(\mathbb{R})$. Thus the Paley–Wiener Theorem applies: G has the form

$$G(k) = \frac{1}{2}\int_{-N}^{N} f(x)e^{ikx}\, dx,$$

where $f\in L_2(-N,N)$. Since G is even, we must also have that $f(-x) = f(x)$, which in turn implies that $G(k) = \int_0^N f(x)\cos kx\, dx$. In other words, $F(z) = G(z^{1/2}) = \int_0^N f(x)\cos\sqrt{z}x\, dx$, as desired.

To prove the converse inclusion, we first claim that

$$\inf_{\lambda\in\mathbb{R}}\left|\frac{E(\lambda)}{E_0(\lambda)}\right| > 0, \tag{4.4}$$

where E_0 is the de Branges function for zero potential: $E_0(z) = \cos\sqrt{z}N - i\sqrt{z}\sin\sqrt{z}N$. To establish (4.4), it clearly suffices to show that

$$\liminf_{\lambda\to\pm\infty}\left|\frac{E(\lambda)}{E_0(\lambda)}\right| > 0.$$

Consider first the case $\lambda\to\infty$, and put again $\lambda = k^2$, $k\to\infty$. Assume that, contrary to the assertion, there exists a sequence $k_n\to\infty$ so that $E(k_n^2)/E_0(k_n^2)\to 0$. We now use (4.3) and the analogous estimate on u' which reads

$$|u'(x,z) + \sqrt{z}\sin\sqrt{z}x| \\ \leqslant \exp(\|V\|_{L_1(0,N)})\exp(|\mathrm{Im}\, z^{1/2}|x) \quad (0\leqslant x\leqslant N). \tag{4.5}$$

Since $E(z) = u(N,z) + iu'(N,z)$, we obtain

$$\left|\frac{E(k_n^2)}{E_0(k_n^2)}\right|^2 = \frac{(\cos k_nN + O(k_n^{-1}))^2 + (k_n\sin k_nN + O(1))^2}{\cos^2 k_nN + k_n^2\sin^2 k_nN}. \tag{4.6}$$

If $k_n\sin k_nN$ remains bounded as $n\to\infty$, then $|\cos k_nN|\to 1$, so (4.6) shows that $|E(k_n^2)/E_0(k_n^2)|^2$ is bounded away from zero. Thus, by passing to a subsequence if necessary, we may assume that $k_n|\sin k_nN|\to\infty$. But then we see from (4.6) that $|E(k_n^2)/E_0(k_n^2)|^2 \to 1$, which is a contradiction to our choice of k_n.

The argument for $\lambda \to -\infty$ is similar (in fact, easier). Write $\lambda = -\kappa^2$ with $\kappa \to \infty$. One shows that both E and E_0 are of the asymptotic form

$$|E(-\kappa^2)|^2 = \frac{\kappa^2}{4} e^{2\kappa N} + O(\kappa e^{2\kappa N}),$$
$$|E_0(-\kappa^2)|^2 = \frac{\kappa^2}{4} e^{2\kappa N} + O(\kappa e^{2\kappa N}),$$

so $|E(-\kappa^2)/E_0(-\kappa^2)| \to 1$. Thus (4.4) holds.

Now if $F(z) = \int f(x) \cos \sqrt{z}x \, dx$ with $f \in L_2(0, N)$, then $F/E_0 \in L_2(\mathbb{R})$, hence also $F/E \in L_2(\mathbb{R})$ by (4.4). Moreover, F is obviously entire.

It remains to establish one of the conditions of Proposition 2.1. To this end, we establish Cauchy type representations for F/E, $F^{\#}/E$. As we have already seen in the proof of Proposition 2.1, such representations imply condition (a) from the proposition.

Write $z = R^2 e^{2i\varphi}$, $\sqrt{z} = Re^{i\varphi}$ with $R > 0$, $0 \leqslant \varphi \leqslant \pi/2$. Then the asymptotic formulae (4.3) and (4.5) yield

$$E(z) = \cos(NRe^{i\varphi}) - iRe^{i\varphi} \sin(NRe^{i\varphi}) + O(e^{NR \sin \varphi}).$$

The constant implicit in the error term is of course independent of R and φ. It follows that

$$|E(z)| \geqslant R|\sin(NRe^{i\varphi})| - O(e^{NR \sin \varphi}). \tag{4.7}$$

Hence there exist constants $C_0, R_0 > 0$ with the following property: If $R \geqslant R_0$ and $\sin \varphi \geqslant C_0/R$, then

$$|E(z)| \geqslant \tfrac{1}{2} Re^{NR \sin \varphi}. \tag{4.8}$$

In the opposite case of small φ, we restrict our attention to the radii $R_n = N^{-1}(2\pi n + \pi/2)$, with $n \in N$, n large. The assumption $\sin \varphi < C_0/R$ ensures that the error term from (4.7) is actually bounded, and

$$\begin{aligned}
\sin(NRe^{i\varphi}) &= \sin(NR \cos \varphi + iNR \sin \varphi) \\
&= \sin(NR + iNR \sin \varphi) + O(R^{-1}) \\
&= \sin(\pi/2 + iNR \sin \varphi) + O(R^{-1}) \\
&= \cosh(NR \sin \varphi) + O(R^{-1}).
\end{aligned}$$

As $\cosh x \geqslant 1$, we thus get from (4.7) that $|E(z)| \geqslant R_n/2$ for z as above and sufficiently large n. Obviously, if $\sin \varphi < C_0/R$, then $e^{-NC_0} e^{NR \sin \varphi} \leqslant 1$, so (4.8), possibly with $1/2$ replaced by a smaller constant, actually holds for all $\varphi \in [0, \pi/2]$ if R is restricted to the values R_n from above. Finally,

$$|F(R^2 e^{2i\varphi})| \leqslant \int_0^N |f(x)| \, |\cos(Re^{i\varphi} x)| \, dx \leqslant \|f\|_2 \sqrt{N} e^{NR \sin \varphi}.$$

In conclusion, it follows that $|F(z)/E(z)| \leqslant CR^{-1}$ for $R \in \{R_n\}$ and $\varphi \in [0, \pi/2]$, and this estimate indeed implies the Cauchy formula

$$\frac{F(z)}{E(z)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} \quad (z \in \mathbb{C}^+)$$

by a standard procedure. (Integrate from $-R_n$ to R_n on the real line and close by a semicircle in the upper half plane; let $n \to \infty$. The above estimate ensures that the integral over the semicircle vanishes in the limit.) The same argument works for $F^{\#}/E$. As explained above, this completes the proof. □

The proof of Theorem 4.2 will depend on an asymptotic formula for the Titchmarsh–Weyl m-function. This subject has been studied systematically and in considerable depth (see, for example, [2,15,17,19,27]). We will only need a rather straightforward result whose proof we include for the reader's convenience.

Given $V \in L_1(0, N)$, we extend V to $(0, \infty)$ by setting it equal to zero on (N, ∞). Denote the spectral measure and the m-function of the half line problem (as usual, with Neumann boundary conditions $y'(0) = 0$) by $\rho^{(N)}$ and $m^{(N)}$, respectively. We now briefly review some basic facts about $m^{(N)}$ and $\rho^{(N)}$; this material can be found, for example, in [10,21]. We can obtain $m^{(N)}$ as follows: Let $f(x, k)$ be the Jost solution. In other words, f solves (3.1) with $z = k^2$ and $f(x, k) = e^{ikx}$ for $x \geqslant N$. Define a meromorphic function M_N by

$$M_N(k) = -\frac{f(0, k)}{f'(0, k)}.$$

Then, since $f(\cdot, k) \in L_2(0, \infty)$ for $k \in \mathbb{C}^+$, we have that $m^{(N)}(k^2) = M_N(k)$ for these k. More precisely, this formula gives the meromorphic continuation of the function $m^{(N)}$, which is originally defined on $\mathbb{C}^+$, to $\mathbb{C} \backslash [0, \infty)$. The m-function $m^{(N)}$ has only finitely many poles in this region. They all lie on $(-\infty, 0)$, and they are just the eigenvalues of $-d^2/dx^2 + V(x)$ on $L_2(0, \infty)$. For $k > 0$, the limit $\lim_{\varepsilon \to 0+} m^{(N)}(k^2 + i\varepsilon)$ exists. We will denote this limit simply by $m^{(N)}(k^2)$; we then have that $m^{(N)}(k^2) = M_N(k)$ for all $k > 0$ (note, however, that $M_N(-k)$ does *not* give the correct value but the complex conjugate of $m^{(N)}(k^2)$ because k^2 is now approached from the lower half plane).

From these facts, we immediately get the following description of $\rho^{(N)}$. Denote the finitely many negative eigenvalues by $-\kappa_n^2$, $\kappa_n > 0$. Then

$$\rho^{(N)} = \sum_n \rho^{(N)}(\{-\kappa_n^2\}) \delta_{-\kappa_n^2} + \frac{1}{\pi} \chi_{(0,\infty)}(\lambda) \mathrm{Im}\, M_N(\sqrt{\lambda})\, d\lambda.$$

We will also need the m-function m_0 and the spectral measure ρ_0 for zero potential. The following formulae hold:

$$m_0(z) = (-z)^{-1/2}, \quad \rho_0 = \chi_{(0,\infty)}(\lambda) \frac{d\lambda}{\pi\sqrt{\lambda}}. \tag{4.9}$$

In the first equation, which holds for $z \in \mathbb{C}^+$, the square root must be chosen so that $\operatorname{Im} m_0 > 0$. Clearly, m_0 can then be holomorphically continued to $\mathbb{C} \setminus [0, \infty)$. This continuation will also be denoted by m_0. Finally, just as for $m^{(N)}$, we put $m_0(\lambda) \equiv \lim_{\varepsilon \to 0+} m_0(\lambda + i\varepsilon)$ for $\lambda > 0$.

Lemma 4.3. (a) *The limit* $\lim_{k \to 0} k M_N(k)$ *exists.*

(b) *For* $\operatorname{Im} k \geq 0$, $k \notin (-\infty, 0]$, *we have that*

$$m^{(N)}(k^2) - m_0(k^2) = \frac{1}{k^2} \int_0^N V(x) e^{2ikx}\, dx + O(|k|^{-3}).$$

Proof. (a) If y_1, y_2 both solve (3.1), then the Wronskian $y_1' y_2 - y_1 y_2'$ is constant. By computing the Wronskian of $f(\cdot, k)$ and $f(\cdot, -k)$ at $x = 0$ and $x = N$, we therefore see that

$$f'(0,k) f(0,-k) - f(0,k) f'(0,-k) = 2ik.$$

Take the derivative with respect to k (writing $\cdot \equiv \frac{d}{dk}$) and then set $k = 0$. We obtain

$$f(0,0) \dot{f}'(0,0) - \dot{f}(0,0) f'(0,0) = i,$$

and thus it is not possible that $f'(0,k)$ and $\dot{f}'(0,k)$ vanish simultaneously at $k = 0$. Therefore a possible pole of M_N at $k = 0$ must be of order one.

(b) Put $g(x,k) = f(x,k) e^{-ikx}$. Then, basically by the variation of constants formula, g is the unique solution of the integral equation

$$g(x,k) = 1 + \frac{1}{2ik} \int_x^N \left(e^{2ik(t-x)} - 1\right) V(t) g(t,k)\, dt.$$

If $\operatorname{Im} k \geq 0$ and $|k| \geq 2\|V\|_{L_1(0,N)}$, this implies the a priori estimate $\|g\|_\infty \leq 2$. So for these k, we have $|g(x,k) - 1| \leq 2\|V\|_1 / |k|$. This in turn shows that

$$g'(x,k) = -\int_x^N V(t) e^{2ik(t-x)}\, dt + O(|k|^{-1}).$$

Hence for large $|k|$,

$$\begin{aligned} m^{(N)}(k^2) = M_N(k) &= -\frac{f(0,k)}{f'(0,k)} = \frac{-g(0,k)}{ikg(0,k) + g'(0,k)} \\ &= \frac{i}{k} \frac{1}{1 + \frac{g'(0,k)}{ikg(0,k)}} = \frac{i}{k}\left(1 - \frac{g'(0,k)}{ikg(0,k)} + O(|k|^{-2})\right) \\ &= \frac{i}{k} + \frac{1}{k^2} \int_0^N V(t) e^{2ikt}\, dt + O(|k|^{-3}), \end{aligned}$$

as desired, since $m_0(k^2) = i/k$. For small k, there is nothing to prove. □

Proof of Theorem 4.2. Suppose that an $F \in S_N$ is given and write, according to Theorem 4.1,

$$F(z) = \int_0^N f(t) \cos \sqrt{z} t \, dt$$

with $f \in L_2(0,N)$. Introduce the (signed) Borel measure σ_N by $\sigma_N = \rho^{(N)} - \rho_0$. Theorem 3.2(b) allows us to compute the norm of F as

$$\begin{aligned} \|F\|_{S_N}^2 = \int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho^{(N)}(\lambda) &= \int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho_0(\lambda) \\ &+ \int_{\mathbb{R}} |F(\lambda)|^2 \, d\sigma_N(\lambda). \end{aligned} \tag{4.10}$$

The two integrals in this last expression converge absolutely because the map $f \mapsto F$ is unitary from $L_2(0,\infty)$ onto $L_2(\mathbb{R}, d\rho_0)$—in fact, it is just the U from (3.3) for zero potential. This observation also says that

$$\int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho_0(\lambda) = \int_0^N |f(t)|^2 \, dt.$$

It remains to analyze the last integral from (4.10). Using the identity

$$\cos x \cos y = \tfrac{1}{2}(\cos(x-y) + \cos(x+y)),$$

we can write it in the form

$$\begin{aligned} &\int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\sigma_N(\lambda) \\ &= \int_{-\infty}^{\infty} d\sigma_N(\lambda) \int_0^N ds \int_0^N dt \overline{f(s)} f(t) \cos \sqrt{\lambda} s \cos \sqrt{\lambda} t \\ &= \frac{1}{2} \int_{-\infty}^{\infty} d\sigma_N(\lambda) \int_0^N ds \int_0^N dt \overline{f(s)} f(t) (\cos \sqrt{\lambda}(s-t) \\ &\quad + \cos \sqrt{\lambda}(s+t)). \end{aligned} \tag{4.11}$$

Formally, this is of the desired form with $\phi(x) = \int \cos \sqrt{\lambda} x \, d\sigma_N(\lambda)$, but this needs to be interpreted carefully because the integral "defining" ϕ will not, in general, be absolutely convergent.

Our strategy will be to first define ϕ as a distribution and then prove that it is actually an absolutely continuous function. More precisely, the contribution coming from $\lambda \in (0,\infty)$ will be treated in this way.

So we define a tempered distribution $\phi_+ \in \mathcal{S}'$ as follows. Let g be a test function from the Schwartz space $\mathcal{S}$. Recall that this means that g is infinitely differentiable and $\sup_{x \in \mathbb{R}} |x|^m |g^{(n)}(x)| < \infty$ for all $m, n \in \mathbb{N}_0$. Then ϕ_+ acts on g by

$$(\phi_+, g) = \int_0^{\infty} d\sigma_N(\lambda) \int_{-\infty}^{\infty} dx \, g(x) \cos \sqrt{\lambda} x.$$

This is well defined because $\int g(x)\cos\sqrt{\lambda}x\,dx$ is rapidly decreasing in λ and from Lemma 4.3 and the preceding material we have the a priori estimate $|\sigma_N|([0,R])\leqslant C\sqrt{R}$. Thus the integral certainly converges. It is also clear that ϕ_+ is linear and continuous in the topology of $\mathscr{S}$, so indeed $\phi_+\in\mathscr{S}'$. Note that formally, ϕ_+ is just $\phi_+(x)=\int_0^\infty\cos\sqrt{\lambda}x\,d\sigma_N(\lambda)$.

The Fourier transform of ϕ_+ is, by definition, the tempered distribution $\hat{\phi}_+$ acting on test functions g by $(\hat{\phi}_+,g)=(\phi_+,\hat{g})$. We compute

$$\begin{aligned}
(\phi_+,\hat{g}) &= \frac{1}{2}\int_0^\infty d\sigma_N(\lambda)\int_{-\infty}^\infty dx\,\hat{g}(x)(e^{i\sqrt{\lambda}x}+e^{-i\sqrt{\lambda}x})\\
&= \sqrt{\frac{\pi}{2}}\int_0^\infty d\sigma_N(\lambda)(g(\sqrt{\lambda})+g(-\sqrt{\lambda}))\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \operatorname{Im}(m^{(N)}-m_0)(k^2)(g(k)+g(-k))k\,dk\\
&= \sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty g(k)|k|\operatorname{Im}(m^{(N)}-m_0)(k^2)\,dk,
\end{aligned}$$

and hence $\hat{\phi}_+$ is a function and

$$\hat{\phi}_+(k)=\sqrt{\frac{2}{\pi}}|k|\operatorname{Im}(m^{(N)}-m_0)(k^2). \tag{4.12}$$

From Lemma 4.3 and the formula for m_0, we see that $\hat{\phi}_+$ is continuous and $\hat{\phi}_+(k)=O(|k|^{-1})$ for large $|k|$. In fact, we get the more precise information that

$$\begin{aligned}
\hat{\phi}_+(k) &= \sqrt{\frac{2}{\pi}}\frac{1}{|k|}\int_0^N V(x)\sin 2|k|x\,dx+O(k^{-2})\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{ik}\int_0^N V(x)(e^{2ikx}-e^{-2ikx})\,dx+O(k^{-2})\\
&= \frac{1}{ik}\widehat{W}_N(k)+O(k^{-2}),
\end{aligned}$$

where

$$W_N(x)=\begin{cases} -(1/2)V(x/2), & 0<x<2N,\\ (1/2)V(-x/2), & -2N<x<0,\\ 0, & |x|>2N.\end{cases}$$

Therefore the (distributional) derivative ϕ'_+ of ϕ_+ has a Fourier transform of the form

$$\widehat{(\phi'_+)}(k)=ik\hat{\phi}_+(k)=\widehat{W}_N(k)+\hat{R}_N(k),$$

where $\hat{R}_N$ is a continuous function and $\hat{R}_N(k) = O(|k|^{-1})$. It follows that

$$\phi'_+(x) = W_N(x) + R_N(x),$$

with $R_N \in L_2$.

In particular, $\phi'_+ \in \mathscr{S}'$ is a locally integrable function, and as a consequence, ϕ_+ is an absolutely continuous function. We define

$$\begin{aligned}\phi(x) &= \int_{-\infty}^{0} \cos\sqrt{\lambda}x \, d\sigma_N(\lambda) + \phi_+(x)\\ &= \sum \rho^{(N)}(\{-\kappa_n^2\}) \cosh \kappa_n x + \phi_+(x),\end{aligned}$$

and we verify that this ϕ has the desired properties.

We know already that ϕ_+ is absolutely continuous. Its Fourier transform, $\hat{\phi}_+$, is real valued and even (cf. (4.12)), so ϕ_+ has these properties, too. The (finite) sum $\sum \rho^{(N)}(\{-\kappa_n^2\}) \cosh \kappa_n x$ manifestly is a smooth, real valued, even function, so we have established that ϕ is absolutely continuous, real valued, and even.

To show that $\phi(0) = 0$, we use the formula

$$(m^{(N)} - m_0)(k^2) = \int_{-\infty}^{\infty} \frac{d\sigma_N(\lambda)}{\lambda - k^2} \quad (k \in \mathbb{C}^+),$$

which follows at once from the Herglotz representations of $m^{(N)}$ and m_0 (see, e.g., [21]). We can Fourier transform the denominator,

$$\frac{1}{\lambda - k^2} = \frac{i}{k} \int_0^{\infty} \cos\sqrt{\lambda}t \, e^{ikt} \, dt \quad (k \in \mathbb{C}^+, \lambda > 0),$$

to write this in the form

$$\begin{aligned}&(m^{(N)} - m_0)(k^2)\\ &\quad = \sum \frac{\rho^{(N)}(\{-\kappa_n^2\})}{-\kappa_n^2 - k^2} + \frac{i}{k} \int_0^{\infty} d\sigma_N(\lambda) \int_0^{\infty} dt \, e^{ikt} \cos\sqrt{\lambda}t.\end{aligned} \tag{4.13}$$

We now take a closer look at this last integral:

$$\begin{aligned}&\int_0^{\infty} d\sigma_N(\lambda) \int_0^{\infty} dt \, e^{ikt} \cos\sqrt{\lambda}t\\ &\quad = \frac{2}{\pi} \int_0^{\infty} dl \, l \, \mathrm{Im}(m^{(N)} - m_0)(l^2) \int_0^{\infty} dt \, e^{ikt} \cos lt\\ &\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dl \, \hat{\phi}_+(l) \int_0^{\infty} dt \, e^{ikt} \cos lt\\ &\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dl \, \hat{\phi}_+(l) \int_0^{\infty} dt \, e^{ikt} e^{-ilt}.\end{aligned}$$

This last expression equals $\int \hat{\phi}_+ \hat{h}$, where $h(t) = \chi_{(0,\infty)}(t) e^{ikt}$. Since we are assuming that $k \in \mathbb{C}^+$, this function is in L_2, as is $\hat{\phi}_+$, and thus we may use the Plancherel identity to obtain the final result

$$(m^{(N)} - m_0)(k^2) = \sum \frac{\rho^{(N)}(\{-\kappa_n^2\})}{-\kappa_n^2 - k^2} + \frac{i}{k}\int_0^\infty \phi_+(t)e^{ikt}\,dt \quad (k \in \mathbb{C}^+).$$

Note that on a formal level, this may be derived very easily from (4.13) because the last term of (4.13) looks like ϕ_+ applied to $(i/k)h$. However, h is not a test function!

If we assume that in addition $\operatorname{Im} k > \max \kappa_n$, then $\int_0^\infty \phi(t)e^{ikt}\,dt$ exists and we get the more compact formula

$$(m^{(N)} - m_0)(k^2) = \frac{i}{k}\int_0^\infty \phi(t)e^{ikt}\,dt.$$

We now specialize to $k = iy$, $y \to \infty$, and integrate by parts. This gives

$$m^{(N)}(-y^2) - m_0(-y^2) = \frac{\phi(0)}{y^2} + \frac{1}{y^2}\int_0^\infty \phi'(t)e^{-yt}\,dt.$$

Since $\phi'_+ \in L_1 + L_2$, the integral goes to zero by dominated convergence, hence

$$m^{(N)}(-y^2) - m_0(-y^2) = \frac{\phi(0)}{y^2} + o(y^{-2}) \quad (y \to \infty).$$

On the other hand, Lemma 4.3(b) implies that $m^{(N)}(-y^2) - m_0(-y^2) = o(y^{-2})$. Therefore, $\phi(0) = 0$.

Let $\mathscr{K}_\phi : L_2(0,N) \to L_2(0,N)$ be the integral operator defined in (4.1), with the ϕ constructed above. We still have to establish the crucial property of ϕ, namely, the fact that the integral from (4.11) equals $\langle f, \mathscr{K}_\phi f\rangle$ for all $f \in L_2(0,N)$.

We first consider the case when $f \in C_0^\infty(0,N)$, and we treat explicitly only the first term from (4.11), which contains $\cos\sqrt{\lambda}(s-t)$. Introduce the new variables $R = s+t$, $r = s-t$. Then we have

$$\begin{aligned}
&\int_{-\infty}^\infty d\sigma_N(\lambda)\int_0^N\int_0^N ds\,dt\,\overline{f(s)}f(t)\cos\sqrt{\lambda}(s-t)\\
&\quad = \frac{1}{2}\int_{-\infty}^\infty d\sigma_N(\lambda)\int_{-\infty}^\infty dr\cos\sqrt{\lambda}r\int_{-\infty}^\infty dR\,\overline{f\left(\frac{R+r}{2}\right)}f\left(\frac{R-r}{2}\right)\\
&\quad = \int_{-\infty}^\infty d\sigma_N(\lambda)\int_{-\infty}^\infty dr\,g(r)\cos\sqrt{\lambda}r.
\end{aligned}$$

Here, we have put

$$g(r) \equiv \frac{1}{2}\int_{-\infty}^\infty dR\,\overline{f\left(\frac{R+r}{2}\right)}f\left(\frac{R-r}{2}\right). \tag{4.14}$$

Note that $g \in C_0^\infty(-N,N)$. In particular, g is an admissable test function, and thus the following manipulations are justified:

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\sigma_N(\lambda) \int_{-\infty}^{\infty} dr\, g(r) \cos\sqrt{\lambda} r \\
&\quad = \sum \rho(\{-\kappa_n^2\}) \int_{-\infty}^{\infty} g(r) \cosh \kappa_n r\, dr \\
&\qquad + \int_0^{\infty} d\sigma_N(\lambda) \int_{-\infty}^{\infty} dr\, g(r) \cos\sqrt{\lambda} r \\
&\quad = \int_{-\infty}^{\infty} dr\, g(r) \sum \rho(\{-\kappa_n^2\}) \cosh \kappa_n r + \int_{-\infty}^{\infty} \phi_+(r) g(r)\, dr \\
&\quad = \int_{-\infty}^{\infty} \phi(r) g(r)\, dr.
\end{aligned}$$

Finally, we can write out g (see (4.14)) and transform back to the original variables (s,t); we obtain the expression

$$\int_0^N \int_0^N ds\, dt\, \overline{f(s)} f(t) \phi(s-t).$$

If we combine this with the result of the analogous computation for the term involving $\cos\sqrt{\lambda}(s+t)$, then we get indeed that

$$\begin{aligned}
&\frac{1}{2} \int_{-\infty}^{\infty} d\sigma_N(\lambda) \int_0^N ds \int_0^N dt\, \overline{f(s)} f(t) (\cos\sqrt{\lambda}(s-t) + \cos\sqrt{\lambda}(s+t)) \\
&\quad = \frac{1}{2} \int_0^N \int_0^N ds\, dt\, \overline{f(s)} f(t) (\phi(s-t) + \phi(s+t)) \\
&\quad = \langle f, \mathscr{K}_\phi f \rangle.
\end{aligned}$$

Using this in (4.10) and (4.11), we see that

$$\|F\|_{S_N}^2 = \langle f, (1 + \mathscr{K}_\phi) f \rangle_{L_2(0,N)}, \tag{4.15}$$

as desired. So far, this has been proved for $f \in C_0^\infty(0,N)$. To establish (4.15) in full generality, fix $f \in L_2(0,N)$ and pick $f_n \in C_0^\infty(0,N)$ with $\|f_n - f\|_{L_2(0,N)} \to 0$. From the proof of Theorem 4.1 (see, in particular, (4.4)) we know that there is a constant $C > 0$ so that for all $G \in S_N$, the inequality $\|G\|_{S_N} \leqslant C\|G\|_{S_N^{(0)}}$ holds, where $S_N^{(0)}$ is the de Branges space for zero potential. Hence, writing $F_n(z) = \int f_n(t) \cos\sqrt{z} t\, dt$, we deduce that

$$\|F_n - F\|_{S_N} \leqslant C\|F_n - F\|_{S_N^{(0)}} = C\|f_n - f\|_{L_2(0,N)} \to 0.$$

Therefore, we can use (4.15) with f replaced by f_n and then pass to the limit to see that (4.15) holds for all $f \in L_2(0,N)$. $\square$

5. The inverse spectral theorem

Theorem 4.2 associates with each Schrödinger equation a function ϕ that determines the scalar product on the corresponding de Branges spaces S_N. Recall also that by Theorem 3.1, these de Branges spaces can be identified with the spaces $L_2(\mathbb{R}, d\rho_N^\beta)$ from the spectral representation of the Schrödinger operators. So it makes sense to think of ϕ (on $[-2N, 2N]$) as representing the spectral data of $-d^2/dx^2 + V(x)$ (on $L_2(0, N)$, with suitable boundary conditions at the endpoints). Our next result is the converse of Theorem 4.2. It says that every function ϕ that has the properties stated in Theorem 4.2 comes from a Schrödinger equation. To be able to formulate this concisely, we denote this set of ϕ's by Φ_N, so

$$\Phi_N = \{\phi : [-2N, 2N] \to \mathbb{R} : \phi \text{ absolutely continuous, even,}\\ \phi(0) = 0, 1 + \mathcal{K}_\phi > 0\}.$$

The last condition of course refers to the integral operator $\mathcal{K}_\phi$ on $L_2(0, N)$ that was introduced in (4.1); we require that the self-adjoint operator $1 + \mathcal{K}_\phi$ be positive definite. In the situation of Theorem 4.2, this condition holds because $\langle f, (1 + \mathcal{K}_\phi) f \rangle$ is a norm.

Theorem 5.1. *For every $\phi \in \Phi_N$, there exists a $V \in L_1(0, N)$ so that the norm on the de Branges space S_N associated with* (3.1) *is given by*

$$\|F\|_{S_N}^2 = \langle f, (1 + \mathcal{K}_\phi) f \rangle_{L_2(0,N)} \quad (f \in L_2(0, N)).$$

Here, $F(z) = \int f(t) \cos \sqrt{z} t \, dt$, *as in Theorem* 4.1.

We will take up the proof of Theorem 5.1 in Section 9. Let us first point out that we also have uniqueness in both directions. In fact, uniqueness is, as usual, much easier than existence.

Theorem 5.2. (a) *If $V \in L_1(0, N)$ is given, then the $\phi \in \Phi_N$ from Theorem* 4.2 *is unique*.

(b) *If $\phi \in \Phi_N$ is given, then the $V \in L_1(0, N)$ from Theorem* 5.1 *is unique*.

This will also be proved in Section 9. We need some preparations; this will occupy us for the following three sections.

6. Canonical systems I

A canonical system is a family of differential equations of the following form:

$$Ju'(x) = zH(x)u(x). \tag{6.1}$$

Here, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $H(x) \in \mathbb{R}^{2\times 2}$, the entries of H are integrable functions on an interval $(0, N)$, and $H(x) \geqslant 0$ (i.e., $H(x)$ is a positive semidefinite matrix) for almost every $x \in (0, N)$. We also assume that there is no nonempty open interval $I \subset (0, N)$ so that $H = 0$ almost everywhere on I. Finally, $z \in \mathbb{C}$ is the spectral parameter.

As usual, $u : [0, N] \to \mathbb{C}^2$ is called a solution if u is absolutely continuous and satisfies (6.1) almost everywhere.

Usually, one does not assume that $H(x) \not\equiv 0$ on nonempty open sets, but dropping this assumption does not add generality. Indeed, by letting

$$S_0 = \{x \in (0, N) : \exists \varepsilon > 0 : H(t) = 0 \text{ for a.e. } t \in (x - \varepsilon, x + \varepsilon)\}$$

and introducing the new independent variable

$$\xi(x) = \int_0^x (1 - \chi_{S_0}(t))\, dt,$$

one may pass to an equivalent canonical system that satisfies our additional assumption.

A fundamental result (namely, Theorem 7.3) associates with every de Branges space a canonical system (6.1). Therefore, canonical systems are a central object in the theory of de Branges spaces.

Let $u(x, z)$, $v(x, z)$ be the solutions of (6.1) with the initial values $u(0, z) = \binom{1}{0}$, $v(0, z) = \binom{0}{1}$. We will mainly work with $u(x, z)$. Just as in Section 3, we can build a de Branges function from u by defining $E_N(z) = u_1(N, z) + iu_2(N, z)$. Here, a pathological case can occur: if $H(x) = \begin{pmatrix} 0 & 0 \\ 0 & H_{22}(x) \end{pmatrix}$ on $(0, N)$, then $u(N, z) = \binom{1}{0}$ and $E_N(z) \equiv 1$. According to our definition in Section 2, this is not a de Branges function. So it will be convenient to slightly extend this definition and to also admit non-zero constants as de Branges functions. The corresponding de Branges space is simply defined to be the zero space.

Proposition 6.1. *$E_N(z) = u_1(N, z) + iu_2(N, z)$ is a de Branges function. The corresponding reproducing kernel J_z is given by*

$$J_z(\zeta) = \int_0^N u^*(x, z) H(x) u(x, \zeta)\, dx.$$

Proof. The formula for J_z follows by a calculation, which is analogous to the discussion preceding Theorem 3.1. One uses the fact that $u(x, \bar{z}) = \overline{u(x, z)}$; we leave the details to the reader. Also, just as in Section 3, by taking $z = \zeta \in \mathbb{C}^+$, the formula for J_z implies that E_N is a de Branges function. In this context, observe the following fact: if

$$\int_0^N u^*(x,z)H(x)u(x,z)\,dx = 0$$

for some $z\in\mathbb{C}$, then $H(x)u(x,z)=0$ for almost every $x\in(0,N)$, hence $u(x,z)=u(0,z)=\binom{1}{0}$. This in turn implies that $H_{11}=0$ almost everywhere, and we are in the trivial case $E_N(z)\equiv 1$. In the opposite case, $\int_0^N u^*(x,z)H(x)u(x,z)\,dx>0$ for all $z\in\mathbb{C}$, and E_N is a genuine de Branges function. $\square$

Eventually, we will again identify the corresponding de Branges space $B(E_N)$ with a space $L_2(\mathbb{R}, d\rho_N^\beta)$, where ρ_N^β is a spectral measure of (6.1), just as we did in the case of Schrödinger equations in Theorem 3.1. However, things are more complicated now, basically for two reasons: First of all, if (6.1) is to be interpreted as an eigenvalue equation $Tu=zu$, then, formally, the operator T should be $Tu=H^{-1}Ju'$, but $H(x)$ need not be invertible. Consequently, one has to work with relations instead of operators. Second, on the so-called singular intervals, Eq. (6.1) actually is a difference equation in disguise. These points will be studied in some detail in Section 10. Our discussion of canonical systems will be modelled on the (simpler) analysis of Section 3. For a functional analytic treatment of canonical systems, see [18]. Ref. [9] also contains a lot of material on canonical systems, though in somewhat implicit form.

7. Four theorems of de Branges

In this section we state, without proof, four general results of de Branges on de Branges spaces which will play an important role in our treatment of the inverse problem for Schrödinger operators. The first result is a useful tool for recognizing de Branges spaces. It is Theorem 23 of [9]. For an alternate proof, see [12, Section 6.1].

Theorem 7.1. *Let $\mathcal{H}$ be a Hilbert space whose elements are entire functions. Suppose that $\mathcal{H}$ has the following three properties*:

(a) *For every $z\in\mathbb{C}$, point evaluation $F\mapsto F(z)$ is a bounded linear functional.*
(b) *If $F\in\mathcal{H}$ has a zero at $w\in\mathbb{C}$, then $G(z)=\frac{z-\bar{w}}{z-w}F(z)$ belongs to $\mathcal{H}$ and $\|G\|=\|F\|$.*
(c) *$F\mapsto F^\#$ is an isometry on $\mathcal{H}$.*

Then $\mathcal{H}$ is a de Branges space: *There exists a de Branges function E, so that $\mathcal{H}=B(E)$ and $\|F\|_{\mathcal{H}}=\|F\|_{B(E)}$ for all $F\in\mathcal{H}$.*

The converse of Theorem 7.1 is also true (and easily proved): Every de Branges space satisfies (a)–(c). In fact, in [5], the conditions of Theorem 7.1 are used to define de Branges spaces.

The de Branges function E is not uniquely determined by the Hilbert space $B(E)$. The situation is clarified by de Branges [5, Theorem I]. Given E, we introduce the entire functions A, B by

$$A(z) = \frac{E(z) + E^{\#}(z)}{2}, \quad B(z) = \frac{E(z) - E^{\#}(z)}{2i}.$$

Theorem 7.2. *Let E_1 and E_2 be de Branges functions. Then $B(E_1) = B(E_2)$ (as Hilbert spaces) if and only if there exists $T \in \mathbb{R}^{2\times 2}$, $\det T = 1$, so that*

$$\begin{pmatrix} A_2(z) \\ B_2(z) \end{pmatrix} = T \begin{pmatrix} A_1(z) \\ B_1(z) \end{pmatrix}.$$

The next two results lie much deeper. They are central to the whole theory of de Branges spaces. We will not state the most general versions here; for this, the reader should consult [9]. The following definition will be useful to avoid (for us) irrelevant technical problems. A de Branges space $B(E)$ is called *regular* if

$$F(z) \in B(E) \Rightarrow \frac{F(z) - F(z_0)}{z - z_0} \in B(E). \tag{7.1}$$

Here z_0 is any fixed complex number. The definition is reasonable because it can be shown that if (7.1) holds for one $z_0 \in \mathbb{C}$, then it holds for all $z_0 \in \mathbb{C}$ (cf. [9, Theorem 25]). Condition (7.1) also plays an important role in [9]. According to (a more general version of) Theorem 7.3, every de Branges space comes from a (possibly singular) canonical system; the regular spaces are precisely those that come from regular problems, that is, $x = 0$ is not a singular endpoint. Jumping ahead, we can also remark that condition (7.1) ensures the existence of a conjugate mapping. See also [31] for other aspects of (7.1).

Theorem 7.3. *If $B(E)$ is a regular de Branges space, $E(0) = 1$, and $N > 0$ is given, then there exists a canonical system* (6.1) (*that is, there exists an integrable function $H : (0, N) \to \mathbb{R}^{2\times 2}$ with $H(x) \geqslant 0$ almost everywhere, $H \not\equiv 0$ on nonempty open sets*), *such that $E(z) = E_N(z)$, where E_N is determined from* (6.1) *as in Proposition* 6.1.

Moreover, $H(x)$ can be chosen so that $\operatorname{tr} H(x)$ is a (positive) constant.

De Branges proved various results of this type; see [6, Theorems V and VII; 9, Theorems 37 and 40]. The version given here follows by combining [6, Theorem VII] with [9, Theorem 27]. In fact, this is not literally true because de Branges uses the equation

$$y(b)J - y(a)J = z\int_a^b y(t)dm(t) \tag{7.2}$$

instead of (6.1). Here, m is a matrix valued measure. If m is absolutely continuous, $dm(t) = m'(t)\,dt$, then (7.2) can be written as a differential equation $y'J = zyH$, with $H = m'$, and the further change of variable $u(x,z) = y^*(x, -\bar{z})$ then gives (6.1). In [9], m is only assumed to be continuous, but then one can change the independent variable to $\xi(t) = \operatorname{tr} m((0,t))$ to get an absolutely continuous measure. This transformation automatically leads to a system with $\operatorname{tr} H(x) \equiv 1$, and this is how one proves the last statement of Theorem 7.3. A further transformation of the type $\xi \to a\xi$ with a suitable $a > 0$ then yields a problem on $(0, N)$ again.

There is no apparent reason for preferring one of these equivalent ways of writing canonical systems (see (6.1) and(7.2)), but it appears that the form we use here (namely (6.1)) has become the most common.

The assumption that $E(0) = 1$ is just a normalization; it does not restrict the applicability of Theorem 7.3. In fact, one can just use Theorem 7.2 with $T = \begin{pmatrix} A(0)|E(0)|^{-2} & B(0)|E(0)|^{-2} \\ -B(0) & A(0) \end{pmatrix}$ to pass to an equivalent E with $E(0) = 1$. This will always work because de Branges functions associated with regular spaces do not have zeros on the real line.

Theorem 7.3, combined with the material from Section 10 (especially (10.5)), is the promised (extremely) general version of the Paley-Wiener Theorem. One can also view Theorem 7.3 as a basic result in inverse spectral theory: given "spectral data" in the form of a de Branges function E, the theorem asserts the existence of a corresponding differential equation. In this paper, we will use Theorem 7.3 in this second way.

As a final remark on Theorem 7.3, we would like to point out that $H(x)$ is uniquely determined by $E(z)$ and N if one normalizes appropriately. (One may require that $\operatorname{tr} H(x)$ be constant, as in the last part of Theorem 7.3, and $\int_0^\varepsilon H_{11}(t)\,dt > 0$ for all $\varepsilon > 0$.) To prove this, the basic idea is to proceed as in the proof of Theorem 5.2 (b) (which will be discussed in Section 9), but things are more complicated here and one needs material from Section 10. We do not need this uniqueness statement in this paper.

Theorem 7.4. *Let $B(E)$, $B(E_1)$, $B(E_2)$ be regular de Branges spaces and assume that $B(E_1)$ and $B(E_2)$ are isometrically contained in $B(E)$. Then either $B(E_1)$ is isometrically contained in $B(E_2)$ or $B(E_2)$ is isometrically contained in $B(E_1)$.*

This is a special case of [9, Theorem 35]. See also [2, Section 6.5] for a proof.

Theorem 7.4 clearly is a strong structural result. The de Branges subspaces of a given space are totally ordered by inclusion. As an illustration, take $B(E) = S_N$, the de Branges space coming from a Schrödinger equation on the interval $(0, N)$. Then it can be deduced from Theorem 7.4 that the chain of spaces $\{S_x : 0 \leqslant x \leqslant N\}$ is a complete list of the de Branges spaces that are subspaces of S_N.

8. Canonical systems II

Theorem 7.3 associates a canonical system to every (regular) de Branges space. Conversely, we have seen in Sections 3 and 6 how Schrödinger equations and canonical systems generate de Branges spaces. This recipe works for other equations as well (Dirac, Sturm–Liouville, Jacobi difference equation). So, in a sense, canonical systems are the most general formally symmetric, second order differential systems (here, by "order" we mean order of differentiation times number of components). In particular, every Schrödinger equation can be written as canonical system by a simple transformation. Namely, given a Schrödinger equation (3.1), let y_1, y_2 be the solutions of (3.1) with $z = 0$ with the initial values $y_1(0) = y_2'(0) = 1$, $y_1'(0) = y_2(0) = 0$, and put $T(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$. Now if $y(x,z)$ solves (3.1), then the vector function u defined by $u(x,z) = T^{-1}(x)\begin{pmatrix} y(x,z) \\ y'(x,z) \end{pmatrix}$ solves (6.1) with

$$H(x) = \begin{pmatrix} y_1^2(x) & y_1(x)y_2(x) \\ y_1(x)y_2(x) & y_2^2(x) \end{pmatrix}.$$

This is shown by direct computation. Note that this H has the required properties: its entries are integrable (in fact, they have absolutely continuous derivatives), and $H(x) \geqslant 0$, $H(x) \neq 0$ for every x.

Conversely, from a canonical system of this special form, one can go back to a Schrödinger equation. We state this separately for later use. By $AC^{(n)}[0, N]$ we denote the set of (locally integrable) functions whose nth derivative (in the sense of distributions) is in $L_1(0, N)$. Equivalently, $f \in AC^{(n)}[0, N]$ precisely if $f, f', \ldots, f^{(n-1)}$ are absolutely continuous on $[0, N]$.

Proposition 8.1. *Let $h, k \in AC^{(2)}[0, N]$ be real valued functions, and suppose that $h(0) = 1$, $h'(0) = 0$, and*

$$h(x)k'(x) - h'(x)k(x) = 1. \tag{8.1}$$

Let

$$H(x) = \begin{pmatrix} h^2(x) & h(x)k(x) \\ h(x)k(x) & k^2(x) \end{pmatrix}.$$

Then, if $u(x,z)$ *solves* (6.1) *with this* H, *then*

$$y(x,z) := h(x)u_1(x,z) + k(x)u_2(x,z)$$

solves (3.1) *with* $V(x) = h''(x)k'(x) - h'(x)k''(x)$. *Moreover, the de Branges spaces generated by* (3.1) *and* (6.1) *as in Section* 3 *and Proposition* 6.1, *respectively, are identical.*

Proof. The fact that y solves (3.1) with $V = h''k' - h'k''$ is checked by direct computation. Note that $hk'' = h''k$; this follows by differentiating (8.1). Also, $hu_1' + ku_2' = 0$ and thus

$$y'(x,z) = h'(x)u_1(x,z) + k'(x)u_2(x,z).$$

In particular, this relation shows that $y(\cdot,z) \in AC^{(2)}[0,N]$.

To compare the de Branges spaces, we must specialize to the solution u with the initial values $u_1(0,z) = 1$, $u_2(0,z) = 0$. The corresponding y satisfies $y(0,z) = 1$, $y'(0,z) = 0$, and hence is the solution from which the de Branges function of the Schrödinger equation is computed. The values at $x = N$ are related by

$$\begin{pmatrix} y(N,z) \\ y'(N,z) \end{pmatrix} = \begin{pmatrix} h(N) & k(N) \\ h'(N) & k'(N) \end{pmatrix} u(N,z).$$

The final claim now follows from Theorem 7.2. $\square$

9. Starting the proofs

Proof of Theorem 5.2. (a) If we know V, we can solve the Schrödinger equation (3.1) (in principle, that is) and find $E_N(z)$. This function in turn determines the scalar products $[F,G]_{S_N}$, and we have that

$$[F,G]_{S_N} = \langle f, (1 + \mathcal{K}_\phi)g \rangle_{L_2(0,N)},$$

so we know the operator $\mathcal{K}_\phi$ on $L_2(0,N)$. Hence we know the kernel $K(s,t)$ almost everywhere on $[0,N] \times [0,N]$ (with respect to two-dimensional Lebesgue measure), but K is continuous, so we actually know the kernel everywhere, and $\phi(2t) = 2K(t,t)$, so ϕ on $[0,2N]$ is uniquely determined by V on $(0,N)$. As ϕ is even, we of course automatically know ϕ on $[-2N,2N]$ then.

(b) Suppose that we have two potentials $V_1, V_2 \in L_1(0,N)$, for which the scalar product on the corresponding de Branges spaces is determined by one and the same $\phi \in \Phi_N$. In other words, $S_N^{(1)} = S_N^{(2)}$ (as de Branges spaces). Now ϕ on $[-2N,2N]$ determines the de Branges spaces $S_x^{(i)}$ $(i = 1,2)$ for every $x \in (0,N]$, so we actually have that also $S_x^{(1)} = S_x^{(2)}$ for these x. By

Theorem 7.2,

$$\begin{pmatrix} u_2(x,z) \\ u_2'(x,z) \end{pmatrix} = T(x) \begin{pmatrix} u_1(x,z) \\ u_1'(x,z) \end{pmatrix} \qquad (0<x\leqslant N),$$

where $T(x)\in\mathbb{R}^{2\times 2}$, $\det T(x)=1$. Comparison of the large z asymptotics with the help of (4.3) and (4.5) shows that $T_{11}=1$, $T_{12}=0$, so $u_1=u_2$. As $V_i(x)=u_i''(x,0)/u_i(x,0)$, this of course implies that $V_1=V_2$. $\square$

We now begin the proof of Theorem 5.1. It is rather obvious how to get started. Let $\phi\in\Phi_N$ be given. If the theorem is true, then, by Theorem 4.1, the spaces

$$H_x=\left\{F(z)=\int_0^x f(t)\cos\sqrt{z}t\,dt : f\in L_2(0,x)\right\}, \tag{9.1}$$

endowed with the scalar products

$$[F,G]_{H_x}=\langle f,(1+\mathscr{K}_\phi)g\rangle_{L_2(0,x)}, \tag{9.2}$$

must be de Branges spaces for $0<x\leqslant N$. This can be confirmed right away.

Lemma 9.1. *H_x with the scalar product $[\cdot,\cdot]_{H_x}$ is a regular de Branges space. The de Branges function E_x for which $H_x=B(E_x)$ may be chosen so that $E_x(0)=1$.*

If $0<x\leqslant y\leqslant N$, then H_x is isometrically contained in H_y.

Proof. We will use Theorem 7.1. H_x obviously is a linear space consisting of entire functions. $[\cdot,\cdot]_{H_x}$ is a scalar product because $1+\mathscr{K}_\phi>0$ (strictly speaking, we know this for the operator on $L_2(0,N)$, but $\langle f,(1+\mathscr{K}_\phi)g\rangle_{L_2(0,x)}$ for $f,g\in L_2(0,x)$ can of course also be evaluated in the bigger space $L_2(0,N)$). $\mathscr{K}_\phi$ is compact, so we actually have that $1+\mathscr{K}_\phi\geqslant\delta>0$. Thus

$$\delta\|f\|^2_{L_2(0,x)}\leqslant\|F\|^2_{H_x}\leqslant C\|f\|^2_{L_2(0,x)},$$

and now completeness of H_x follows from the completeness of $L_2(0,x)$.

We now verify conditions (a)–(c) of Theorem 7.1. Condition (a) is obvious from

$$|F(z)|\leqslant e^{|z|^{1/2}x}\int_0^x |f(t)|\,dt\leqslant x^{1/2}e^{|z|^{1/2}x}\|f\|\leqslant(x/\delta)^{1/2}e^{|z|^{1/2}x}\|F\|_{H_x}.$$

It is also clear that (c) holds since $F^{\#}(z)=\int\overline{f(t)}\cos\sqrt{z}t\,dt$, so $F^{\#}\in H_x$ and, as K is real valued,

$$\|F^{\#}\|^2=\langle\bar{f},(1+\mathscr{K}_\phi)\bar{f}\rangle=\langle f,(1+\mathscr{K}_\phi)f\rangle=\|F\|^2.$$

To prove (b), fix $w \in \mathbb{C}$ and $F \in H_x$ with $F(w) = 0$. Extend the $f \in L_2(0, x)$ corresponding to F to $(-x, x)$ by letting $f(-t) = f(t)$ $(0 < t < x)$. Then

$$F(k^2) = \frac{1}{2} \int_{-x}^{x} f(t) e^{-ikt} \, dt = \sqrt{\frac{\pi}{2}} \hat{f}(k).$$

The function $\hat{f}$ is entire, even, obeys $|\hat{f}(k)| \leqslant Ce^{x|k|}$, its restriction to $\mathbb{R}$ belongs to $L_2(\mathbb{R})$ and $\hat{f}(\pm\sqrt{w}) = 0$. Put

$$\hat{g}(k) = \frac{k^2 - \bar{w}}{k^2 - w} \hat{f}(k).$$

Then $\hat{g}$ is also entire, $|\hat{g}(k)| \leqslant Ce^{x|k|}$, and the restriction of $\hat{g}$ to $\mathbb{R}$ is square integrable. Hence the Paley–Wiener Theorem applies: There exists $g \in L_2(-x, x)$ so that

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} g(t) e^{-ikt} \, dt.$$

Since $\hat{g}$ is even, g must also be even. It follows that

$$\frac{k^2 - \bar{w}}{k^2 - w} F(k^2) = \sqrt{\frac{\pi}{2}} \hat{g}(k) = \frac{1}{2} \int_{-x}^{x} g(t) e^{-ikt} \, dt = \int_0^x g(t) \cos kt \, dt,$$

and hence the function $G(z) = \frac{z - \bar{w}}{z - w} F(z)$ is of the form

$$G(z) = \int_0^x g(t) \cos \sqrt{z} t \, dt,$$

with $g \in L_2(0, x)$, so $G \in H_x$. We now calculate the norm of G. In this calculation, we extend ϕ to a function on $\mathbb{R}$ by setting it equal to zero outside $[-2x, 2x]$. We do this in order to have a well-behaved Fourier transform $\hat{\phi}$. Note that $\mathscr{K}_\phi$, viewed as an operator on $L_2(0, x)$, does not depend on the values of $\phi(t)$ for $|t| > 2x$. We also rewrite $\langle g, (1 + \mathscr{K}_\phi) g \rangle_{L_2(0,x)}$. Namely, since g is even and the integral kernel K of $\mathscr{K}_\phi$ satisfies

$$K(s, t) = K(-s, t) = K(s, -t) = K(-s, -t),$$

we have that

$$\langle \chi_{(0,x)} g, \mathscr{K}_\phi \chi_{(0,x)} g \rangle_{L_2(0,x)} = \frac{1}{8} \int_{-x}^{x} ds \int_{-x}^{x} dt \, \overline{g(s)} g(t) (\phi(s - t) + \phi(s + t)).$$

Furthermore, using the substitution $s \to -s$ in the second term, we can write this in the form

$$\begin{aligned} \langle \chi_{(0,x)} g, \mathscr{K}_\phi \chi_{(0,x)} g \rangle_{L_2(0,x)} &= \frac{1}{4} \int_{-x}^{x} ds \int_{-x}^{x} dt \, \overline{g(s)} g(t) \phi(s - t) \\ &= \frac{1}{4} \langle g, \phi * g \rangle_{L_2(-x,x)}, \end{aligned}$$

where, as usual, the star denotes convolution. Having made these preliminary observations and using the fact that $|\hat{f}(k)| = |\hat{g}(k)|$ for real k, we obtain

$$\begin{aligned}
4\|G\|_{H_x}^2 &= 4\langle \chi_{(0,x)}g, (1+\mathscr{K}_\phi)\chi_{(0,x)}g\rangle_{L_2(0,x)}\\
&= 2\|g\|_{L_2(-x,x)}^2 + \langle g, \phi * g\rangle_{L_2(-x,x)}\\
&= 2\|\hat{g}\|_{L_2(\mathbb{R})}^2 + \langle \hat{g}, \hat{\phi}\,\hat{g}\rangle_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} |\hat{g}(k)|^2(2+\hat{\phi}(k))\,dk\\
&= \int_{-\infty}^{\infty} |\hat{f}(k)|^2(2+\hat{\phi}(k))\,dk\\
&= 2\|\hat{f}\|_{L_2(\mathbb{R})}^2 + \langle \hat{f}, \hat{\phi}\,\hat{f}\rangle_{L_2(\mathbb{R})} = 2\|f\|_{L_2(-x,x)}^2 + \langle f, \phi * f\rangle_{L_2(-x,x)}\\
&= 4\langle \chi_{(0,x)}f, (1+\mathscr{K}_\phi)\chi_{(0,x)}f\rangle_{L_2(0,x)} = 4\|F\|_{H_x}^2,
\end{aligned}$$

as desired. Theorem 7.1 now shows that H_x is a de Branges space.

It is clear that for every $\lambda \in \mathbb{R}$, there exists an $F \in H_x$ with $F(\lambda) \neq 0$. Thus, by the definition of de Branges spaces, the corresponding de Branges function cannot have zeros on the real line. Using Theorem 7.2, we can therefore normalize so that $E_x(0) = 1$ (exactly as in the remark following Theorem 7.3).

We still must show that the de Branges space H_x is regular. We will check condition (7.1) with $z_0 = 0$. So let $F \in H_x$, $F(z) = \int_0^x f(t)\cos\sqrt{z}t\,dt$, with $f \in L_2(0,x)$. Then

$$g(t) := \int_t^x f(s)(t-s)\,ds$$

is in $AC^{(2)}[0,x]$ (so, in particular, $g \in L_2(0,x)$), and $g'(t) = \int_t^x f(s)\,ds$, $g'' = -f$. By integrating by parts twice, we thus see that

$$\begin{aligned}
\int_0^x g(t)\cos\sqrt{z}t\,dt &= \frac{\sin\sqrt{z}t}{\sqrt{z}} \int_t^x f(s)(t-s)\,ds\Bigg|_{t=0}^{t=x}\\
&\quad - \int_0^x dt\,\frac{\sin\sqrt{z}t}{\sqrt{z}} \int_t^x ds\, f(s)\\
&= \frac{\cos\sqrt{z}t}{z}\int_t^x f(s)\,ds\Bigg|_{t=0}^{t=x} + \int_0^x f(t)\frac{\cos\sqrt{z}t}{z}\,dt\\
&= \frac{F(z)-F(0)}{z},
\end{aligned}$$

hence this latter combination is in H_x.

The final claim of the lemma is obvious from the construction of the spaces H_x. We have made this statement explicit mainly because of its importance. $\square$

The next step in the proof of Theorem 5.1 is to apply Theorem 7.3 to the regular de Branges space $H_N = B(E_N)$. We obtain $H : (0,N) \to \mathbb{R}^{2\times 2}$, with entries in $L_1(0,N)$ and $H(x) \geqslant 0$ for almost every $x \in (0,N)$, $\operatorname{tr} H(x) = \tau > 0$, such that $E_N(z)$ is exactly the de Branges function associated with canonical system (6.1) as in Proposition 6.1.

Actually, we have obtained much more. We get a whole scale of de Branges spaces in both cases. On the one hand, we have the spaces $H_x = B(E_x)$ from Lemma 9.1. On the other hand, we can consider canonical system (6.1) on $(0, x)$ only; by Proposition 6.1, we get again de Branges spaces, which we denote by B_x. Our next major goal is to show that, possibly after a reparametrization of the independent variable, $H_x = B_x$ for all $x \in [0, N]$ (at the moment, we know this only for $x = N$). One crucial input will be Theorem 7.4; however, we will also need additional material on canonical systems. This topic will be resumed in the next section.

The proof of Theorem 5.1 will then proceed as follows. The identity $H_x = B_x$ says that we have two realizations of the same chain of de Branges spaces: one from Lemma 9.1 and a second one from canonical system (6.1). By comparing objects in these two worlds, we will get information on the matrix elements $H_{ij}(x)$ of the H from (6.1). This will allow us to verify the hypotheses of Proposition 8.1; so the spaces H_x we started with indeed come from a Schrödinger equation.

10. Canonical systems III

We now develop some material on the spectral representation of canonical systems. We consider Eq. (6.1) together with the boundary conditions

$$u_2(0) = 0, \qquad u_1(N)\cos\beta + u_2(N)\sin\beta = 0. \tag{10.1}$$

Here, $\beta \in [0, \pi)$. As usual, z is called an eigenvalue if there is a nontrivial solution to (6.1) and (10.1). We can considerably simplify the whole discussion by excluding certain "singular" values of β. In particular, it is convenient to assume right away that $\beta \neq \pi/2$. Then zero is not an eigenvalue.

In particular, the following holds. If $f \in L_1(0, N)$ is given, then the inhomogeneous problem $Ju' = f$ together with boundary conditions (10.1) has a unique solution u which can be written in the form

$$u(x) = \int_0^N G(x,t) f(t)\, dt,$$

$$G(x,t) = \begin{pmatrix} \tan\beta & -\chi_{(0,t)}(x) \\ -\chi_{(0,x)}(t) & 0 \end{pmatrix} = G(t,x)^*.$$

We can now write eigenvalue problem (6.1), (10.1) as an integral equation, which is easier to handle. Of course, this is a standard procedure; compare, for example, [16, Chapter VI]. Let $L_2^H(0, N)$ be the space of measurable functions $y:(0,N)\to\mathbb{C}^2$ satisfying

$$\|y\|^2_{L_2^H(0,N)} \equiv \int_0^N y^*(x)H(x)y(x)\,dx < \infty.$$

The quotient of $L_2^H(0,N)$ by $\mathcal{N} = \{y : \|y\| = 0\}$ is a Hilbert space. As usual, this space will again be denoted by $L_2^H(0,N)$, and we will normally not distinguish between Hilbert space elements and their representatives. In a moment, we will also use the similarly defined space L_2^I, where H is replaced by the 2×2 identity matrix. The space L_2^I can be naturally identified with $L_2 \oplus L_2$.

As a preliminary observation, notice that a nontrivial solution y to (6.1) and (10.1) cannot be the zero element of $L_2^H(0,N)$. Indeed, if $\|y\|_{L_2^H} = 0$, then $H(x)y(x) = 0$ almost everywhere, so (6.1) implies that $y(x) = y(0)$. But since $\beta \neq \pi/2$, the boundary conditions (10.1) then force $y \equiv 0$. A similar argument shows that eigenfunctions associated with different eigenvalues also represent different elements of $L_2^H(0,N)$.

We now claim that λ is an eigenvalue of (6.1) and (10.1) with corresponding eigenfunction y if and only if $y \in L_2^H(0,N)$ and y solves

$$y(x) = \lambda \int_0^N G(x,t)H(t)y(t)\,dt. \tag{10.2}$$

Note that for $y \in L_2^H(0,N)$, (10.2) may be considered in the pointwise sense or as an equation in $L_2^H(0,N)$. Fortunately, the two interpretations are equivalent. More precisely, if (10.2) holds in $L_2^H(0,N)$, then we can simply define a particular representative $y(x)$ by the right-hand side of (10.2) (this right-hand side does not depend on the choice of representative).

It is clear from the construction of G and the fact that solutions of (6.1) are continuous that eigenfunctions lie in $L_2^H(0,N)$ and solve (10.2) pointwise. Conversely, if $y \in L_2^H(0,N)$, then $Hy \in L_1(0,N)$. So if y in addition solves (10.2), then it also solves (6.1) and (10.1) by construction of G again.

Now define a map

$$V : L_2^H(0,N) \to L_2^I(0,N), \quad y(x) \mapsto H^{1/2}(x)y(x).$$

Here, $H^{1/2}(x)$ is the unique positive semidefinite square root of $H(x)$. In the sequel, we will often use the fact that $H(x)$ and $H^{1/2}(x)$ have the same kernel. V is an isometry and hence maps L_2^H unitarily onto its range $R(V) \subset L_2^I$. Define an integral operator $\mathcal{L}$ on $L_2^I(0,N)$ by

$$L(x,t) = H^{1/2}(x)G(x,t)H^{1/2}(t),$$

$$(\mathcal{L}f)(x) = \int_0^N L(x,t)f(t)\,dt.$$

The kernel L is square integrable (by this we mean that $\int_0^N \int_0^N \|L^*L\|\,dx\,dt < \infty$), so $\mathcal{L}$ is a Hilbert–Schmidt operator and thus compact. Since $L(x,t) = L^*(t,x)$, $\mathscr{L}$ is also self-adjoint. The following lemma says that the eigenvalues of (6.1) and (10.1) are precisely the reciprocal values of the nonzero eigenvalues of $\mathscr{L}$. The corresponding eigenfunctions are mapped to one another by applying V.

Lemma 10.1. *Let $f \in L_2^I(0,N)$, $\lambda \neq 0$. Then the following statements are equivalent*:

(a) $\mathscr{L}f = \lambda^{-1}f$;
(b) $f \in R(V)$, *and the unique* $y \in L_2^H(0,N)$ *with* $Vy = f$ *solves* (10.2).

Proof. Note that for all $g \in L_2^I$, we have that $(\mathscr{L}g)(x) = H^{1/2}(x)w(x)$, where

$$w(x) = \int_0^N G(x,t)H^{1/2}(t)g(t)\,dt$$

lies in L_2^H, thus $R(\mathscr{L}) \subset R(V)$. Now if (a) holds, then $f = \lambda\mathscr{L}f \in R(V)$, so $f = Vy$ for a unique $y \in L_2^H$ and

$$f(x) = H^{1/2}(x)y(x) = \lambda(\mathscr{L}Vy)(x) = \lambda H^{1/2}(x)\int_0^N G(x,t)H(t)y(t)\,dt$$

for almost every $x \in (0,N)$. In other words,

$$H^{1/2}(x)\left(y(x) - \lambda\int_0^N G(x,t)H(t)y(t)\,dt\right) = 0$$

almost everywhere, and this says that the expression in parantheses is the zero element of L_2^H, that is, (10.2) holds.

Conversely, if (b) holds, we only need to multiply (10.2) from the left by $H^{1/2}(x)$ to obtain (a). $\square$

Let $P : L_2^I \to L_2^I$ be the projection onto the closed subspace $R(V)$ of L_2^I. Since

$$R(V)^{\perp} = \{f \in L_2^I : H(x)f(x) = 0 \text{ almost everywhere}\},$$

we have that $\mathscr{L}(1-P) = 0$. Also, we have already observed that $R(\mathscr{L}) \subset R(V) = R(P)$, so $\mathscr{L} = P\mathscr{L}$. Hence $\mathscr{L}P = P\mathscr{L}$, and thus $R(P) = R(V)$ is a reducing subspace for $\mathscr{L}$. Let $\mathscr{L}_0 : R(V) \to R(V)$ be the restriction of $\mathscr{L}$ to $R(V)$. Then $\mathscr{L}_0$ is also compact (in fact, Hilbert–Schmidt) and self-adjoint, and $\mathscr{L} = \mathscr{L}_0 \oplus 0$.

If we use this notation, then Lemma 10.1 says that the eigenfunctions of $\mathscr{L}_0$ with nonzero eigenvalues precisely correspond to the eigenfunctions of (6.1) and (10.1). The kernel of $\mathscr{L}_0$ will also play a central role. To develop this, we now introduce two important subspaces of L_2^H. Namely, let

$$R_{(0,N)} = \{y \in L_2^H(0,N) : \exists f \in AC^{(1)}[0,N], H(x)f(x) = 0 \text{ for a.e. } x \in (0,N), \\ f_2(0) = 0, f(N) = 0, Jf' = Hy\},$$

$$\tilde{R}_{(0,N)} = \{y \in L_2^H(0,N) : \exists f \in AC^{(1)}[0,N], H(x)f(x) = 0 \text{ for a.e. } x \in (0,N), \\ f_2(0) = 0, f_1(N)\cos\beta + f_2(N)\sin\beta = 0, Jf' = Hy\}.$$

Recall that on a formal level, operators T associated with (6.1) should act as $Tf = H^{-1}Jf'$, so (still formally) the relation $Jf' = Hy$ says that y is an image of f. Thus $\tilde{R}_{(0,N)}$ should be thought of as the space of images of zero; $R_{(0,N)}$ has a similar interpretation. In the following lemma, we identify $\tilde{R}_{(0,N)}$ as the kernel $N(\mathscr{L}_0)$ of $\mathscr{L}_0$.

Lemma 10.2. $N(\mathscr{L}_0) = V\tilde{R}_{(0,N)}$.

Proof. If $g \in R(V)$ with $\mathscr{L}_0 g = 0$ is given, write $g(x) = H^{1/2}(x)y(x)$ with $y \in L_2^H$. Then y obeys

$$H^{1/2}(x)\int_0^N G(x,t)H(t)y(t)\,dt = 0 \tag{10.3}$$

in $L_2^I(0,N)$, that is, for almost every $x \in (0,N)$. Let $f(x) = \int_0^N G(x,t)H(t)y(t)\,dt$. Then, by the construction of G, $f \in AC^{(1)}[0,N]$, f satisfies boundary conditions (10.1), and $Jf' = Hy$; by (10.3), $H(x)f(x) = 0$ for almost every $x \in (0,N)$. So $y \in \tilde{R}_{(0,N)}$ and $g = Vy \in V\tilde{R}_{(0,N)}$.

Conversely, suppose that $g = Vy$ for some $y \in \tilde{R}_{(0,N)}$. By definition of $\tilde{R}_{(0,N)}$, there exists $f \in AC^{(1)}[0,N]$, so that $H(x)f(x) = 0$ almost everywhere, f satisfies the boundary conditions and $Jf' = Hy$. We have $\mathscr{L}_0 g = \mathscr{L}Vy = V\tilde{f}$, where $\tilde{f}(x) = \int_0^N G(x,t)H(t)y(t)\,dt$. Again by construction of G, the function $\tilde{f} \in AC^{(1)}[0,N]$ thus solves the following problem: $\tilde{f}$ satisfies the boundary conditions and $J\tilde{f}' = Hy$. However, as noted at the beginning of this section, there is only one function with these properties, hence $\tilde{f} = f$, and therefore $(\mathscr{L}_0 g)(x) = H^{1/2}(x)f(x) = 0$ almost everywhere. $\square$

Theorem 10.3. *Suppose that $\beta \neq \pi/2$. Then the normed eigenfunctions of the boundary value problem* (6.1) *and* (10.1),

$$Jy'(x) = zH(x)y(x), \qquad y_2(0) = 0, \quad y_1(N)\cos\beta + y_2(N)\sin\beta = 0,$$

form an orthonormal basis of the Hilbert space $L_2^H(0,N) \ominus \tilde{R}_{(0,N)}$.

Proof. As $\mathscr{L}_0$ is compact and self-adjoint, the normed eigenfunctions of $\mathscr{L}_0$ (suitably chosen in the case of degeneracies) form an orthonormal basis of $R(V)$. Also, the normed eigenfunctions belonging to nonzero eigenvalues form an orthonormal basis of $R(V) \ominus N(\mathscr{L}_0)$. Now go back to L_2^H, using the unitary map $V^{-1} : R(V) \to L_2^H(0,N)$. By Lemma 10.2, $N(\mathscr{L}_0)$ is mapped onto $\tilde{R}_{(0,N)}$, and by Lemma 10.1, the preceding discussion and the fact that $\mathscr{L} = \mathscr{L}_0 \oplus 0$, the eigenfunctions of $\mathscr{L}_0$ with nonzero eigenvalues precisely go to the eigenfunctions of (6.1) and (10.1). $\square$

As in Section 3, we can introduce spectral measures ρ_N^β. Define

$$\rho_N^\beta = \sum_{\frac{u_1}{u_2}(N,\lambda) = -\tan\beta} \frac{\delta_\lambda}{\|u(\cdot,\lambda)\|^2_{L_2^H(0,N)}}.$$

The sum is over the eigenvalues $\{\lambda_n\}$ of (6.1) and (10.1) (which also depend on N and β). Recall also that $u(\cdot,z)$ is the solution of (6.1) with $u(0,z) = \binom{1}{0}$. The map U defined by

$$U : L_2^H(0,N) \ominus \tilde{R}_{(0,N)} \to L_2(\mathbb{R}, d\rho_N^\beta),$$
$$(Uf)(\lambda) = \int_0^N u^*(x,\lambda) H(x) f(x)\, dx$$

is unitary. Indeed, this is just a reformulation of Theorem 10.3 because U computes the scalar products of f with the elements of the basis $\{u(\cdot,\lambda_n)\}$. The $u(\cdot,\lambda_n)$'s are not normalized here, but this has been taken into account by choosing the correct weights in the definition of ρ_N^β.

For a further development of the theory of canonical systems, we need the following definition. Following [6,9], we call $x_0 \in (0,N)$ a *singular point* if there exists an $\varepsilon > 0$, so that on $(x_0 - \varepsilon, x_0 + \varepsilon)$, H has the form

$$H(x) = h(x) P_\varphi, \quad P_\varphi = \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}$$

for some (x-independent) $\varphi \in [0,\pi)$ and some $h \in L_1(x_0 - \varepsilon, x_0 + \varepsilon)$, $h \geqslant 0$. Notice that P_φ is the projection onto $e_\varphi = \binom{\cos\varphi}{\sin\varphi}$. Points that are not singular are called *regular points*. Clearly, the set S of singular points,

$$S = \{x \in (0,N) : x \text{ is singular}\},$$

is open, so it can be represented as a countable or finite union of disjoint, open intervals:

$$S = \bigcup (a_n, b_n).$$

On such an interval (a_n, b_n), the angle $\varphi = \varphi_n$ whose existence is (locally) guaranteed by the definition above must actually have the same value on the whole interval, for otherwise there would be regular points on (a_n, b_n). We call the boundary condition β at $x = N$ *regular* if $\beta \neq \pi/2$ and, in case there should be an n with $b_n = N$, then also $\beta \neq \varphi_n$, where φ_n is the angle corresponding to the interval (a_n, b_n).

To get a first intuitive understanding of the notion of singular points, consider (6.1) on an interval $(a,b)\subset S$. After multiplying from the left by $J^{-1}=-J$, the equation reads

$$u'(x) = -zh(x)JP_\varphi u(x).$$

Since the matrices on the right-hand side commute with one another for different values of x, the solution is given by

$$u(x) = \exp\left(-z\int_a^x h(t)\,dt\ JP_\varphi\right)u(a).$$

However, $P_\varphi JP_\varphi = 0$, as we see either from a direct computation or alternatively from the fact that this matrix is singular, anti-self-adjoint and has real entries. Thus the series for the exponential terminates and

$$u(x) = \left(1 - z\int_a^x h(t)\,dt\,JP_\varphi\right)u(a).$$

In particular, letting $u_+ = u(b)$, $u_- = u(a)$, $H = \int_a^b H(x)\,dx$, we obtain $J(u_+ - u_-) = zHu_-$, so on a singular interval, (6.1) actually is its difference equation analog in disguise.

Lemma 10.4. *Suppose* $y\in\tilde{R}_{(0,N)}$, *and let* $f\in AC^{(1)}[0,N]$ *be such that* $H(x)f(x)=0$ *almost everywhere*, $f_2(0)=0$, $f_1(N)\cos\beta + f_2(N)\sin\beta = 0$, *and* $Jf' = Hy$ (*the existence of such an* f *follows from the definition of* $\tilde{R}_{(0,N)}$). *Then*, *if* $x_0\in(0,N)$ *is regular*, *then* $f(x_0)=0$. *Similarly*, *if* β *is a regular boundary condition*, *then* $f(N)=0$.

Proof. Fix y, f, x_0 as above, and write $f(x) = R(x)\binom{\sin\varphi(x)}{-\cos\varphi(x)}$. Since $Hf=0$ almost everywhere, either $R(x_0)=0$ or else $H(x)$ must have the form $H(x) = h(x)P_{\varphi(x)}$ in a neighborhood of x_0. (Note that this does not say that x_0 is singular because φ may depend on x.) In the first case, we are done. If $R(x_0)\neq 0$, we can solve for R, φ in terms of f_1, f_2 in a neighborhood of x_0, and we find that these functions are absolutely continuous, too. Hence the condition that $Jf' = Hy$ gives

$$\begin{aligned} R'(x)\begin{pmatrix}\cos\varphi(x)\\ \sin\varphi(x)\end{pmatrix} + R(x)\varphi'(x)\begin{pmatrix}-\sin\varphi(x)\\ \cos\varphi(x)\end{pmatrix} &= h(x)P_{\varphi(x)}y(x)\\ &\equiv \alpha(x)\begin{pmatrix}\cos\varphi(x)\\ \sin\varphi(x)\end{pmatrix}. \end{aligned}$$

We now take the scalar product with $\binom{-\sin\varphi(x)}{\cos\varphi(x)}$ and find that $R(x)\varphi'(x)=0$. Hence $R(x_0)\neq 0$ implies that $\varphi'\equiv 0$ on a neighborhood of x_0, that is, x_0 is singular. This contradiction shows that $f(x_0)=0$.

This argument also works at $x_0 = N$, provided that $(N-\varepsilon, N)\not\subset S$ for all $\varepsilon>0$. On the other hand, if $(N-\varepsilon,N)\subset(a_n,b_n)\subset S$ for some $\varepsilon>0$, then near

N, the function f must have the form $f(x) = R(x)\binom{\sin\varphi_n}{-\cos\varphi_n}$. But now the boundary condition at N implies that $R(N) = 0$ or

$$\sin\varphi_n \cos\beta - \cos\varphi_n \sin\beta = \sin(\varphi_n - \beta) = 0.$$

This latter relation, however, cannot hold if β is regular. $\square$

Here is an immediate consequence of the second part of Lemma 10.4.

Corollary 10.5. *If β is regular, then $\tilde{R}_{(0,N)} = R_{(0,N)}$.*

We can now prove the promised analog of Theorem 3.1.

Theorem 10.6. *For regular boundary conditions β, the Hilbert spaces $L_2(\mathbb{R}, d\rho_N^\beta)$ and $B(E_N)$ (see Proposition 6.1) are identical. More precisely, if $F(z) \in B(E_N)$, then the restriction of F to $\mathbb{R}$ belongs to $L_2(\mathbb{R}, d\rho_N^\beta)$, and $F \mapsto F|_\mathbb{R}$ is a unitary map from $B(E_N)$ onto $L_2(\mathbb{R}, d\rho_N^\beta)$.*

Proof. Basically, we repeat the proof of Theorem 3.1. As β and N are fixed throughout, we will again usually drop the reference to these parameters. Let $\{\lambda_n\}$ be the eigenvalues of (6.1) and (10.1). We claim again that $J_z \in L_2(\mathbb{R}, d\rho)$ for every $z \in \mathbb{C}$ and verify this by the following calculation:

$$\begin{aligned}
\|J_z\|^2_{L_2(\mathbb{R},d\rho)} &= \sum_n |J_z(\lambda_n)|^2 \rho(\{\lambda_n\}) \\
&= \sum_n |\langle u(\cdot,z), u(\cdot,\lambda_n)\rangle_{L_2^H(0,N)}|^2 \|u(\cdot,\lambda_n)\|^{-2}_{L_2^H(0,N)} \\
&\leqslant \|u(\cdot,z)\|^2_{L_2^H(0,N)}.
\end{aligned}$$

The estimate follows with the help of Bessel's inequality. Similar reasoning shows that

$$\langle J_w, J_z \rangle_{L_2(\mathbb{R},d\rho)} = \langle u(\cdot,z), Qu(\cdot,w)\rangle_{L_2^H(0,N)},$$

where Q is the projection onto $\overline{L(\{u(\cdot,\lambda_n)\})}$. By Theorem 10.3 and Corollary 10.5, $\overline{L(\{u(\cdot,\lambda_n)\})} = R_{(0,N)}^\perp$.

We now want to show that $u(\cdot,z) \in R_{(0,N)}^\perp$ for all $z \in \mathbb{C}$. To this end, fix $y \in R_{(0,N)}$, and pick $f \in AC^{(1)}[0,N]$ with $Hf = 0$ almost everywhere,

$f_2(0) = f(N) = 0$, and $Jf' = Hy$. An integration by parts shows

$$\begin{aligned}\langle u(\cdot, z), y \rangle_{L_2^H(0,N)} &= \int_0^N u^*(x,z)H(x)y(x)\,dx = \int_0^N u^*(x,z)Jf'(x)\,dx \\ &= u^*(x,z)Jf(x)|_{x=0}^{x=N} - \int_0^N u'^*(x,z)Jf(x)\,dx \\ &= \int_0^N (Ju'(x,z))^* f(x)\,dx \\ &= \bar{z} \int_0^N u^*(x,z)H(x)f(x)\,dx = 0,\end{aligned}$$

as desired. Thus $Qu(\cdot, w) = u(\cdot, w)$ and

$$\langle J_w, J_z \rangle_{L_2(\mathbb{R}, d\rho)} = \langle u(\cdot, z), u(\cdot, w) \rangle_{L_2^H(0,N)} = J_z(w) = [J_w, J_z]_{B(E_N)}.$$

This discussion of Qu and the use of Bessel's inequality (instead of Parseval's identity) were the only modifications that are necessary; the rest of the argument now proceeds literally as in the proof of Theorem 3.1. $\square$

The observations that were made after the proof of Theorem 3.1 also have direct analogs. By combining Theorem 10.6 with the remarks following Theorem 10.3, we get an induced unitary map, which we still denote by U. It is given by

$$U : L_2^H(0,N) \ominus R_{(0,N)} \to B(E_N) \tag{10.4a}$$

$$(Uf)(z) = \int u^*(x, \bar{z})H(x)f(x)\,dx. \tag{10.4b}$$

The proof goes as in Section 3. One first checks that (10.4b) is correct for $f = u(\cdot, \lambda_n)$. This follows from the following calculation:

$$\begin{aligned}(Uu(\cdot, \lambda_n))(z) &= \int_0^N u^*(x, \bar{z})H(x)u(x, \lambda_n)\,dx \\ &= \int_0^N \overline{u^*(x,z)H(x)u(x,\lambda_n)}\,dx \\ &= \int_0^N (u^*(x,z)H(x)u(x,\lambda_n))^*\,dx \\ &= \int_0^N u^*(x,\lambda_n)H(x)u(x,z)\,dx = J_{\lambda_n}(z).\end{aligned}$$

Then one extends to the whole space. In this context, recall also that $u(\cdot, \lambda_n) \in R_{(0,N)}^{\perp}$, as we saw in the proof of Theorem 10.3.

It is remarkable that the technical complications we have had to deal with in this section are, so to speak, automatically handled correctly by the U from (10.4a) and (10.4b). Namely, first of all, the boundary condition β does not appear in (10.4a) and (10.4b). Recall that above we needed a regular β, but once Theorem 10.6 has been proved, we can get a statement that does not involve β by passing from $L_2(\mathbb{R}, d\rho_N^\beta)$ to the de Branges space $B(E_N)$.

Next, (10.4b) also makes sense for general $f \in L_2^H(0,N)$, not necessarily orthogonal to $R_{(0,N)}$. If interpreted in this way, U is partial isometry from $L_2^H(0,N)$ to $B(E_N)$ with initial space $L_2^H(0,N) \ominus R_{(0,N)}$ and final space $B(E_N)$. This follows again from the fact that $u(\cdot,z) \in R_{(0,N)}^\perp$ for all $z \in \mathbb{C}$.

We can immediately make good use of these observations to prove the following important fact.

Theorem 10.7. *The de Branges spaces* $B(E_N)$ *coming from canonical system* (6.1) (cf. *Proposition* 6.1) *are regular.*

Proof. Again, we prove this by verifying (7.1) for $z_0 = 0$. As a direct consequence of the discussion above, we have that

$$B(E_N) = \left\{ F(z) = \int_0^N u^*(x,\bar{z}) H(x) f(x)\, dx : f \in L_2^H(0,N) \right\}. \tag{10.5}$$

Thus integration by parts yields

$$\begin{aligned}
\frac{F(z)-F(0)}{z} &= \int_0^N \frac{u^*(x,\bar{z}) - (1,0)}{z} H(x) f(x)\, dx \\
&= -\frac{u^*(x,\bar{z}) - (1,0)}{z} \int_x^N H(t) f(t)\, dt \Big|_{x=0}^{x=N} \\
&\quad + \frac{1}{z} \int_0^N dx\, u'^*(x,\bar{z}) \int_x^N dt\, H(t) f(t) \\
&= \int_0^N dx\, u^*(x,\bar{z}) H(x) J \int_x^N dt\, H(t) f(t) \\
&\equiv \int_0^N u^*(x,\bar{z}) H(x) g(x)\, dx,
\end{aligned}$$

with $g(x) = J \int_x^N H(t) f(t)\, dt$. This g is bounded, hence in $L_2^H(0,N)$, so the proof is complete. $\square$

Note that the relation (10.5) also makes it clear why it is reasonable to interpret Theorem 7.3 as a Paley–Wiener Theorem.

We are now heading toward the analog of Theorem 3.2(a). For $0 < N_1 < N_2$, we define

$$\begin{aligned}
R_{(N_1,N_2)} = \{ y \in L_2^H(N_1,N_2) : \exists f \in AC^{(1)}[N_1,N_2], f(N_1) = f(N_2) = 0, \\
H(x) f(x) = 0 \text{ for a.e. } x \in (N_1,N_2), Jf' = Hy \}.
\end{aligned}$$

The desired result (see Corollary 10.9) will be a consequence of the following observation.

Lemma 10.8. *Let* $0<N_1<N_2$, *and suppose that* N_1 *is regular. Then*

$$R_{(0,N_2)} = R_{(0,N_1)} \oplus R_{(N_1,N_2)},$$

$$L_2^H(0,N_2) \ominus R_{(0,N_2)} = (L_2^H(0,N_1) \ominus R_{(0,N_1)}) \oplus (L_2^H(N_1,N_2) \ominus R_{(N_1,N_2)}).$$

Proof. The second equation of course follows from the first one. If $y \in R_{(0,N_2)}$ and $f \in AC^{(1)}[0,N_2]$ is as in the definition of $R_{(0,N_2)}$, then Lemma 10.4 implies that $f(N_1)=0$. This shows that $\chi_{(0,N_1)}y \in R_{(0,N_1)}$ and $\chi_{(N_1,N_2)}y \in R_{(N_1,N_2)}$ because as the required f's we can just take the corresponding restrictions of the original f.

Conversely, if $y = y^{(1)} + y^{(2)}$ with $y^{(1)} \in R_{(0,N_1)}$, $y^{(2)} \in R_{(N_1,N_2)}$, then the $f^{(i)}$'s from the definition satisfy $f^{(1)}(N_1) = f^{(2)}(N_1) = 0$. Hence $f := f^{(1)} + f^{(2)}$ is an absolutely continuous function with the properties needed to deduce that $y \in R_{(0,N_2)}$. □

Corollary 10.9. *Let* $0<N_1<N_2$, *and suppose that* N_1 *is regular. Then* $B(E_{N_1})$ *is isometrically contained in* $B(E_{N_2})$.

Proof. Lemma 10.8 says that

$$L_2^H(0,N_1) \ominus R_{(0,N_1)} \subset L_2^H(0,N_2) \ominus R_{(0,N_2)},$$

the inclusion being isometric. The unitary operator U from (10.4b) maps these spaces onto $B(E_{N_1})$ and $B(E_{N_2})$, respectively. □

We conclude this section with a closer study of the effect of singular points. As above, we first look at the L_2^H spaces and then transfer the results to the scale of de Branges spaces $B(E_N)$ by applying U from (10.4b).

Lemma 10.10. *Let* $0<N_1<N_2$, *and suppose that* N_1 *is regular and* $(N_1,N_2) \subset S$. *Then*

$$\dim(L_2^H(N_1,N_2) \ominus R_{(N_1,N_2)}) = 1,$$

$$L_2^H(0,N_2) \ominus R_{(0,N_2)} = (L_2^H(0,N_1) \ominus R_{(0,N_1)}) \oplus V,$$

where V *is a one-dimensional space.*

Proof. On (N_1,N_2), we have that $H(x) = h(x)P_\varphi$. So, for an arbitrary $y \in L_2^H(N_1,N_2)$, the function $H(x)y(x)$ has the form $H(x)y(x) = h(x)w(x)e_\varphi$, where $\int_{N_1}^{N_2} |w|^2 h < \infty$. Here, as introduced above, $e_\varphi = \binom{\cos\varphi}{\sin\varphi}$. Now if $f \in AC^{(1)}[N_1,N_2]$ obeys $Jf' = Hy$ and $f(N_1)=0$, then

$$f(x) = -Je_\varphi \int_{N_1}^{x} w(t)h(t)\,dt. \tag{10.6}$$

Thus, the additional condition $f(N_2) = 0$ forces

$$\int_{N_1}^{N_2} w(t)h(t)\,dt = 0. \tag{10.7}$$

Conversely, if (10.7) holds, we can define $f \in AC^{(1)}[N_1, N_2]$ by (10.6), and this f satisfies $f(N_1) = f(N_2) = 0$ and $Jf' = Hy$. Moreover, since $e_\varphi^* J e_\varphi = 0$, it follows that $Hf = 0$ on (N_1, N_2).

But the integral from (10.7) is the scalar product in $L_2^H(N_1, N_2)$ of y with the constant function e_φ, so we have proved that

$$R_{(N_1,N_2)} = \{y \in L_2^H(N_1, N_2) : \langle e_\varphi, y \rangle_{L_2^H(N_1,N_2)} = 0\} = \{e_\varphi\}^\perp.$$

As e_φ is not the zero element of $L_2^H(N_1, N_2)$, this is the first assertion. The second claim follows from the first one with the help of Lemma 10.8. (Incidentally, the condition that N_1 be regular is needed only for this implication.) $\square$

Corollary 10.11. *If* $N_1 > 0$ *is regular, but* $(N_1, N_2) \subset S$, *then* $B(E_{N_2}) = B(E_{N_1}) \oplus V$, *where* V *is a one-dimensional space.*

Similarly, if $(0, N) \subset S$, *then either* $B(E_N) = \{0\}$ *or* $B(E_N) \cong \mathbb{C}$. *In the first case,* $E_N(z) \equiv 1$.

Sketch of proof. The first part follows in the usual way from Lemma 10.10 by applying U from (10.4b). The second part is established by a similar discussion; we leave the details to the reader. Let us just point out the fact that the case $B(E_N) = \{0\}$ occurs if $H(x) = h(x)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on $(0, N)$. Here, the boundary condition at zero is not regular, so to speak, and we have that $R_{(0,N)} = \tilde{R}_{(0,N)} = L_2^H(0, N)$. $\square$

11. Matching de Branges spaces

We resume the proof of Theorem 5.1. Recall briefly what we have done already: We have constructed two families of de Branges spaces, H_x and B_x, $0 < x \leqslant N$. The spaces H_x are given by (9.1) and (9.2). The spaces B_x come from a canonical system,

$$Ju'(x) = zH(x)u(x). \tag{11.1}$$

This system is trace normed, that is, $\operatorname{tr} H(x) = \tau > 0$ for all $x \in (0, N)$. It will be convenient to define $H_0 = B_0 = \{0\}$. Canonical system (11.1) was constructed such that $B_N = H_N$; in fact, the corresponding de Branges functions are equal. We also know that H_t is isometrically contained in H_x if $t \leqslant x$, and for the family B_x, we have Corollary 10.9. In particular, if

$x \in [0, N]$ is arbitrary and if $t \in [0, N]$ is a regular value of (11.1), then H_x and B_t are both isometrically contained in $H_N = B_N$. Here, the points $t = 0$ and N are regular by definition; the claim on B_t is obvious for these t's. (In a different context, it would of course make perfect sense to call $t = 0$ singular if there is an interval $(0, s) \subset S$.) Denote this (extended) set of regular values by R, that is, $R = [0, N] \backslash S$.

By Lemma 9.1 and Theorem 10.7, all spaces are regular, so Theorem 7.4 applies: either $H_x \subset B_t$ or $B_t \subset H_x$, the inclusion being isometric in each case. Define, for $t \in R$, a function $x(t)$ by

$$x(t) = \inf\{x \in [0, N] : H_x \supset B_t\}.$$

It is clear that $x(t)$ is increasing, $x(0) = 0$, and since H_x is a proper subspace of $H_N = B_N$ for $x < N$, $x(N) = N$. Our next goal is to prove that $H_{x(t)} = B_t$.

A modification of (11.1) will be useful to avoid certain trivialities. Namely, if $(0, N)$ starts with a singular interval $(0, b) \subset S$ and if $E_b(z) \equiv 1$, we simply delete this initial interval $(0, b)$ (and rescale so that we end up with a problem on $(0, N)$ again). Of course, this does not change the de Branges space B_N. We have just removed an interval on which nothing happens.

Next, we show that the spaces H_x depend continuously on x in the following sense.

Lemma 11.1. *For every $x \in (0, N)$, we have that*

$$H_x = \overline{\bigcup_{t<x} H_t} = \bigcap_{t>x} H_t.$$

In the second expression, the closure is taken in H_N; recall that this space contains H_x as a subspace for every x.

Proof. We begin with the first equality. We know already that H_t is isometrically contained in H_x for $t < x$, and this implies that the closure of the union is contained in H_x. Conversely, let $F \in H_x$, so $F(z) = \int f(s) \cos\sqrt{z}s \, ds$ for some $f \in L_2(0, x)$. Let $F_n(z) = \int_0^{x-1/n} f(s) \cos\sqrt{z}s \, ds$. Then $F_n \in H_{x-1/n}$ and $F_n \to F$ in H_x because

$$\|F - F_n\|_{H_x} \leqslant C\|f - \chi_{(0,x-1/n)} f\|_{L_2(0,x)} \to 0.$$

Thus $F \in \overline{\bigcup_{t<x} H_t}$.

As for the second assertion, the ordering of the spaces H_t here implies that $H_x \subset \bigcap_{t>x} H_t$. On the other hand, if $F \in \bigcap_{t>x} H_t$, then for all large n, we have that $F(z) = \int f_n(s) \cos\sqrt{z}s \, ds$ for some $f_n \in L_2(0, x + 1/n)$. But by the uniqueness of the Fourier transform, there can be at most one function $f \in L_2(\mathbb{R})$ so that $F(z) = \int f(s) \cos\sqrt{z}s \, ds$, hence $f = f_n$ for all n. This f is supported by $(0, x + 1/n)$ for all n, hence $f \in L_2(0, x)$ and $F \in H_x$. $\square$

Proposition 11.2. *The (modified) system* (11.1) *has no singular points, and* $B_t = H_{x(t)}$ *for all* $t \in [0, N]$.

Proof. We first prove that the desired relation $B_t = H_{x(t)}$ holds for all $t \in R$, the set of regular values. For these t, we know that for all x, either $H_x \subset B_t$ or $B_t \subset H_x$. Now the definition of $x(t)$ implies that the first case occurs for $x < x(t)$ and the second inclusion holds for $x > x(t)$. Hence

$$\overline{\bigcup_{x<x(t)} H_x} \subset B_t \subset \bigcap_{x>x(t)} H_x,$$

and now Lemma 11.1 shows that $B_t = H_{x(t)}$. This argument does not literally apply to the extreme values $t = 0$ and N, but the claim is obvious in these cases.

If (a, b) is a component of S, then the preceding may be applied to the regular values a, b, and thus

$$H_{x(b)} \ominus H_{x(a)} = B_b \ominus B_a.$$

Corollary 10.11 shows that this latter difference is one dimensional. (For $a = 0$, this statement holds because of our modification of (11.1).) On the other hand, $H_{x(b)} \ominus H_{x(a)}$ is isomorphic to $L_2(x(a), x(b))$ and hence either the zero space or infinite dimensional. We have reached a contradiction which can only be avoided if $S = \emptyset$. $\square$

It is, of course, much more convenient to have $B_t = H_t$; this can be achieved by transforming (11.1). More specifically, we will use $x(t)$ as the independent variable. We defer the discussion of the technical details to Section 15 because we need additional tools which will be developed next.

12. The conjugate function

In regular de Branges spaces, one can introduce a so-called conjugate mapping, which is a substitute for the Hilbert transform of ordinary Fourier analysis. In this paper, the conjugate mapping will not play a major role. Thus, our treatment of this topic will be very cursory and incomplete; for the full picture, please consult [9].

Consider a canonical system and the associated de Branges spaces $B_N \equiv B(E_N)$; for simplicity, we assume that there are no singular points (as in Proposition 11.2). Recall that v is the solution of (11.1) with $v(0, z) = \binom{0}{1}$, and define

$$K_z(\zeta) = \frac{v^*(N, z)Ju(N, \zeta) - 1}{\zeta - \bar{z}}. \tag{12.1}$$

Since $(v^*(x,z)Ju(x,\zeta))' = (\zeta - \bar{z})v^*(x,z)H(x)u(x,\zeta)$, this may also be written in the form

$$K_z(\zeta) = \int_0^N v^*(x,z)H(x)u(x,\zeta)\,dx = \int_0^N u^*(x,\bar{\zeta})H(x)v(x,\bar{z})\,dx. \quad (12.2)$$

In particular, when combined with (10.5), the last expression shows that $K_z \in B_N$ for all $z \in \mathbb{C}$. We can thus define, for $F \in B_N$, the conjugate function $\tilde{F}$ by $\tilde{F}(z) = [K_z, F]$. The material of Section 10 immediately provides us with an interpretation of $\tilde{F}$. Namely, if $F(z) = \int u^*(x,\bar{z})H(x)f(x)\,dx$ with $f \in L_2^H(0,N)$, then $\tilde{F}(z) = \int v^*(x,\bar{z})H(x)f(x)\,dx$, that is, instead of u, one uses the solution v satisfying the "conjugate" boundary condition at $x = 0$.

The next lemma says that $\tilde{F}$ does not depend on the space in which the conjugate function is computed. Notice that since we are assuming that all points are regular, B_{N_1} is isometrically contained in B_{N_2} for $N_1 < N_2$ by Corollary 10.9.

Lemma 12.1. *Let* $0 < N_1 < N_2$ *and* $F \in B_{N_1}$. *Then*

$$\tilde{F}(z) = [K_z^{(N_1)}, F]_{B_{N_1}} = [K_z^{(N_2)}, F]_{B_{N_2}}.$$

Proof. By (12.2), $K_z^{(N_i)} = U(\chi_{(0,N_i)}v(\cdot,\bar{z}))$, where U is the map from (10.4b). Since there are no singular points, U is unitary from $L_2^H(0,N_i)$ onto B_{N_i}. Also, $F = Uf$ for some $f \in L_2^H(0,N_1)$, hence

$$\begin{aligned}
[K_z^{(N_1)}, F]_{B_{N_1}} &= [U(\chi_{(0,N_1)}v(\cdot,\bar{z})), Uf]_{B_{N_1}} = \langle v(\cdot,\bar{z}), f\rangle_{L_2^H(0,N_1)} \\
&= \langle \chi_{(0,N_2)}v(\cdot,\bar{z}), f\rangle_{L_2^H(0,N_2)} = [U(\chi_{(0,N_2)}v(\cdot,\bar{z})), Uf]_{B_{N_2}} \\
&= [K_z^{(N_2)}, F]_{B_{N_2}}. \qquad \square
\end{aligned}$$

For $F \in B_N$, we introduce the abbreviation

$$(RF)(z) = \frac{F(z) - F(0)}{z}.$$

Recall from Theorem 10.7 that $RF \in B_N$ whenever $F \in B_N$. The following identity, which is a special case of a more general identity (cf. [9, Theorem 27]), is basically about all we will need about the conjugate function.

Proposition 12.2. *For all* $F, G \in B_N$, *the following identity holds*:

$$\tilde{F}(0)\overline{G(0)} - F(0)\overline{\tilde{G}(0)} = [RG, F] - [G, RF].$$

Proof. By a standard density argument, it suffices to prove this for $F = J_{\lambda_m}$, $G = J_{\lambda_n}$, where the $\{\lambda_k\}$ are the eigenvalues of (11.1) and (10.1). In that case,

$$\tilde{F}(0) = [K_0, J_{\lambda_m}] = \overline{K_0(\lambda_m)} = K_0(\lambda_m) = \frac{u_1(N, \lambda_m) - 1}{\lambda_m}$$

and $\tilde{G}(0) = (u_1(N, \lambda_n) - 1)/\lambda_n$. Also, $F(0) = -u_2(N, \lambda_m)/\lambda_m$ and $G(0) = -u_2(N, \lambda_n)/\lambda_n$. So the left-hand side of the equation that we want to prove equals

$$\frac{1}{\lambda_m \lambda_n}[-(u_1(N, \lambda_m) - 1)u_2(N, \lambda_n) + u_2(N, \lambda_m)(u_1(N, \lambda_n) - 1)].$$

Because $u(\cdot, \lambda_m)$ and $u(\cdot, \lambda_n)$ both satisfy the boundary condition at $x = N$, we have that

$$u_1(N, \lambda_m)u_2(N, \lambda_n) = u_1(N, \lambda_n)u_2(N, \lambda_m),$$

and thus the above expression simplifies to

$$\frac{u_2(N, \lambda_n) - u_2(N, \lambda_m)}{\lambda_m \lambda_n}. \tag{12.3}$$

On the other hand,

$$[RG, F] = \overline{(RG)(\lambda_m)} = (RG)(\lambda_m) = \frac{J_{\lambda_n}(\lambda_m) - G(0)}{\lambda_m},$$

and similarly for $[G, RF]$. Since $J_{\lambda_n}(\lambda_m) = \delta_{mn} J_{\lambda_n}(\lambda_n)$, a brief calculation now shows that the right-hand side of the asserted equation also equals (12.3). $\square$

13. The integral equations

In this section, we study the reproducing kernels $J_z^{(x)}$ and the conjugate kernels $K_z^{(x)}$ of the spaces H_x. More specifically, we find integral equations from which further properties of these quantities will be derived later. Actually, it suffices to consider the case when $z = 0$.

Things are very easy for $J_0^{(x)}$. Introduce $y(x, t)$ by writing

$$J_0^{(x)}(z) = \int_0^x y(x, t) \cos\sqrt{z}t \, dt.$$

So for fixed x, the function $y(x, \cdot)$ lies in $L_2(0, x)$. Now we use the defining property of $J_0^{(x)}$: $[J_0^{(x)}, F]_{H_x} = F(0)$ for all $F \in H_x$. Write $F(z) = \int f(t) \cos\sqrt{z}t \, dt$, where $f \in L_2(0, x)$, and recall that by (9.2),

$$[J_0^{(x)}, F]_{H_x} = \langle y(x, \cdot), (1 + \mathscr{K}_\phi)f \rangle_{L_2(0,x)}.$$

We obtain the equation

$$\langle y(x,\cdot),(1+\mathcal{K}_\phi)f\rangle_{L_2(0,x)} = F(0) = \langle 1,f\rangle_{L_2(0,x)},$$

where 1 on the right-hand side denotes the function that is identically equal to 1 on $(0,x)$. The operator $1+\mathcal{K}_\phi$ is self-adjoint and $f\in L_2(0,x)$ is arbitrary, so we conclude that

$$y(x,t)+\int_0^x K(t,s)y(x,s)\,ds = 1, \tag{13.1}$$

and this is the desired integral equation for $y(x,t)$, which, in turn, determines $J_0^{(x)}(z)$. By the way we derived (13.1), this equation is to be interpreted as an equation in $L_2(0,x)$, where $x\in(0,N]$ is arbitrary, but fixed. However, we can then again define a particular representative $y(x,t)$ by requiring (13.1) to hold pointwise. (Recall that the same procedure was applied in connection with (10.2).) From the definition of y we know only that $y(x,\cdot)\in L_2(0,x)$, but then (13.1) in this pointwise sense of course implies much more regularity. This will be discussed in detail in the next section.

As for $K_0^{(x)}$, we would like to proceed similarly, but we must first identify a conjugate mapping $F\mapsto\tilde{F}(0)$ on H_x. We do this by exploiting the fact that the identity of Proposition 12.2 already determines $\tilde{F}(0)$ up to a multiple of $F(0)$.

Let

$$\psi(t)=\int_0^t \phi(s)(t-s)\,ds+t.$$

Then $\psi\in AC^{(3)}[0,N]$, $\psi''=\phi$, and $\psi(0)=0$, $\psi'(0)=1$. For $F\in H_N$, so $F(z)=\int_0^N f(t)\cos\sqrt{z}t\,dt$ with $f\in L_2(0,N)$, we define

$$\hat{F}(0)=\int_0^N f(t)\psi(t)\,dt. \tag{13.2}$$

Proposition 13.1. *For all $F,G\in H_N$, the following identity holds*:

$$\hat{F}(0)\overline{G(0)}-F(0)\overline{\hat{G}(0)}=[RG,F]-[G,RF].$$

Proof. As usual, let f,g be the $L_2(0,N)$ functions associated with F and G, respectively. Plugging in the definitions, we see that the left-hand side of the identity that is to be proved equals

$$\int_0^N\int_0^N dx\,dt\,\overline{g(t)}f(x)\left(x-t+\int_0^x\phi(s)(x-s)\,ds-\int_0^t\phi(s)(t-s)\,ds\right).$$

We have seen in the last part of the proof of Lemma 9.1 that

$$(RF)(z)\equiv\frac{F(z)-F(0)}{z}=\int_0^N dt\cos\sqrt{z}t\int_t^N ds\,f(s)(t-s).$$

Thus, again by a routine calculation, the right-hand side of the above identity is equal to

$$\int_0^N \int_0^N dx\,dt\,\overline{g(t)}f(x)\bigg(x-t+\frac{1}{2}\int_0^t (\phi(s-x)+\phi(s+x))(s-t)\,ds \\ -\frac{1}{2}\int_0^x (\phi(s-t)+\phi(s+t))(s-x)\,ds\bigg).$$

So the identity holds for arbitrary $F, G \in H_N$ precisely if

$$\int_0^x [\phi(t-s)+\phi(t+s)-2\phi(s)](s-x)\,ds \\ = \int_0^t [\phi(x-s)+\phi(x+s)-2\phi(s)](s-t)\,ds \tag{13.3}$$

for all $t, x \in [0, N]$. In other words, we need to show that the function

$$H(x,t) = \int_0^x [\phi(t-s)+\phi(t+s)-2\phi(s)](s-x)\,ds$$

is symmetric: $H(x,t) = H(t,x)$.

We first prove this under the additional assumption that ϕ is smooth (let us say, $\phi \in C^2$). Then H has continuous partial derivatives up to order 2. We have

$$H(0,t) = H_x(0,t) = 0, \quad H_{xx}(x,t) = 2\phi(x) - \phi(t-x) - \phi(t+x),$$

and, because ϕ is even, $H(x,0) = 0$. Moreover,

$$H_t(x,0) = \int_0^x \big[\phi'(-s)+\phi'(s)\big](s-x)\,ds$$

is also equal to zero because the expression in brackets is zero. Finally,

$$H_{tt}(x,t) = \int_0^x [\phi''(t-s)+\phi''(t+s)](s-x)\,ds,$$

and by integrating by parts, we may evaluate this as

$$H_{tt}(x,t) = 2\phi(t) - \phi(t-x) - \phi(t+x).$$

Fix $t \in [0, N]$ and consider the difference $D(x) = H(x,t) - H(t,x)$. What we have shown in the preceding paragraph says that $D(0) = D'(0) = 0$, $D''(x) \equiv 0$. Hence $D \equiv 0$, as desired.

To prove (13.3) in full generality, approximate the given ϕ uniformly on $[-2N, 2N]$ by even functions $\phi_n \in C^2$. (For example, approximate the odd function $\phi' \in L_1(-2N, 2N)$ in L_1 norm by odd functions $f_n \in C_0^\infty(-2N, 2N)$ and let $\phi_n(x) = \int_0^x f_n(t)\,dt$.) Then (13.3) holds for ϕ_n, and we can pass to the limit to obtain (13.3) for ϕ as well. $\square$

We are now ready to introduce $K_0^{(x)}$, the conjugate kernel (for $z = 0$) of H_x. The map $F \mapsto \hat{F}(0)$ is a bounded linear functional on H_x. Hence there exists a unique $K_0^{(x)} \in H_x$, so that $[K_0^{(x)}, F]_{H_x} = \hat{F}(0)$ for all $F \in H_x$. The use of the symbol $K_0^{(x)}$ for this function will be justified in Section 15. More precisely, we will show that, possibly after a modification of the canonical system from Proposition 11.2, $K_0^{(x)}$ indeed also is the conjugate kernel of B_x, which was introduced in Section 12.

Theorem 13.2. *Let $J_0^{(x)}$ and $K_0^{(x)}$ be the reproducing and conjugate kernels, respectively, of H_x. Define $y(x,\cdot), w(x,\cdot) \in L_2(0,x)$ by*

$$J_0^{(x)}(z) = \int_0^x y(x,t) \cos\sqrt{z}t \, dt,$$

$$K_0^{(x)}(z) = \int_0^x w(x,t) \cos\sqrt{z}t \, dt.$$

Then y, w obey the integral equations

$$y(x,t) + \int_0^x K(t,s) y(x,s) \, ds = 1,$$

$$w(x,t) + \int_0^x K(t,s) w(x,s) \, ds = \psi(t).$$

Proof. The assertions concerning $J_0^{(x)}$ and y have been established at the beginning of this section (cf. (13.1)). The argument for $K_0^{(x)}$ and w is completely analogous. By definition of $K_0^{(x)}$, (13.2), and the fact that ψ is real,

$$[K_0^{(x)}, F]_{H_x} = \hat{F}(0) = \langle \psi, f \rangle_{L_2(0,x)}.$$

On the other hand,

$$[K_0^{(x)}, F]_{H_x} = \langle w(x,\cdot), (1 + \mathscr{K}_\phi) f \rangle_{L_2(0,x)} = \langle (1 + \mathscr{K}_\phi) w(x,\cdot), f \rangle_{L_2(0,x)},$$

and as $f \in L_2(0,x)$ is arbitrary, the integral equation follows. $\square$

14. Regularity properties

In this section, we investigate the regularity of the solutions $p(x,t)$ of integral equations of the form

$$p(x,t) + \int_0^x K(t,s) p(x,s) \, ds = g(t). \tag{14.1}$$

Here, K still is the kernel from (4.1). For $g(t) = 1$ and $g(t) = \psi(t)$, (14.1) reduces to the equations from Theorem 13.2. Since $K(t,s) = (\phi(s-t) + \phi(s+t))/2$ and $\phi \in AC^{(1)}$, we expect that the solutions p have similar regularity, at least if g is sufficiently smooth. In fact, more is true: p has better regularity properties than K!

The material of this section is of a technical character. It is possible to omit the proof of the following theorem on a first reading.

To simplify the notation, we introduce the set

$$\Delta_N = \{(x,t) \in \mathbb{R}^2 : 0 \leqslant t \leqslant x \leqslant N\}.$$

By continuity in Δ_N, we will always mean that the function under consideration is jointly continuous in $(x,t) \in \Delta_N$. Note that it is also possible to consider (14.1) for $x = 0$; we then simply have that $p(0,0) = g(0)$.

Theorem 14.1. *Suppose that* $\phi \in \Phi_N$ *and* $g \in AC^{(2)}[0,N]$. *Then, for every* $x \in [0,N]$, *integral equation* (14.1) *has a unique solution* $p(x,\cdot)$ *in* $L_2(0,x)$ *which has the following regularity properties*:

(a) $p \in C^1(\Delta_N)$, *that is, the first order partial derivatives exist and are continuous in* Δ_N.
(b) $p(x,x) \in AC^{(2)}[0,N]$.

For reasons of brevity, our formulation in part (a) is a little sloppy. The easiest way to get a precise statement is to interpret (a) as follows: The first order partial derivatives exist on the interior of Δ_N, and they have continuous extensions to Δ_N. By a limiting argument, this implies that the one-sided partial derivatives exist where they can be reasonably defined.

The second order partial derivatives exist in a certain weak sense. Unfortunately, things get messy (for example, the statements are not symmetric in x and t), and it is better to avoid these issues as much as possible by using an approximation argument as in the proof of Proposition 13.1. Therefore, we have not made these statements explicit. We do need, however, the statement on the existence of the second derivative of the "diagonal" function $p(x,x)$.

Proof. Integral equation (14.1) is of the form

$$(1 + \mathscr{K}_\phi^{(x)})p(x,\cdot) = g,$$

where we write $\mathscr{K}_\phi^{(x)}$ for the integral operator in $L_2(0,x)$ that is generated by the kernel K. Since $1 + \mathscr{K}_\phi^{(x)} > 0$ by assumption, (14.1) has the unique solution

$$p(x,\cdot) = (1 + \mathscr{K}_\phi^{(x)})^{-1} g. \tag{14.2}$$

Note that the roles of the variables x and t are quite different: t is the independent variable, while x is an external parameter.

We now observe the important fact that (14.2) makes sense not only on $L_2(0,x)$, but on each space of the following chain of Banach spaces:

$$C[0,x] \subset L_2(0,x) \subset L_1(0,x).$$

Indeed, first of all, $\mathscr{K}_\phi^{(x)}$ is a well-defined operator on each of these three spaces; in fact, $\mathscr{K}_\phi^{(x)}$ maps $L_1(0,x)$ into $C[0,x]$. Moreover, $\mathscr{K}_\phi^{(x)}$ is compact in every case. This follows from the Arzela–Ascoli Theorem: If $f_n \in L_1(0,x)$, $\|f_n\|_1 \leqslant 1$, then, since the kernel K is uniformly continuous on $[0,x]\times[0,x]$, the sequence of functions $\mathscr{K}_\phi^{(x)} f_n$ is equicontinuous and uniformly bounded, hence there exists a uniformly convergent subsequence. So $\mathscr{K}_\phi^{(x)}$ is compact even as an operator from $L_1(0,x)$ to $C[0,x]$.

The inclusion $\mathscr{K}_\phi^{(x)}(L_1(0,x)) \subset C[0,x]$ also shows that the spectrum of $\mathscr{K}_\phi^{(x)}$ is independent of the space: the eigenfunctions with nonzero eigenvalues are always contained in the smallest space $C[0,x]$. In particular, we always have that $-1 \notin \sigma(\mathscr{K}_\phi^{(x)})$, so $1+\mathscr{K}_\phi^{(x)}$ is boundedly invertible and (14.2) holds.

To investigate the derivatives of p, we again temporarily make the additional assumption that ϕ and g (and hence also K) are smooth. So, let us suppose that $\phi, g \in C^\infty$. Then one can show that the solution p is also smooth: $p \in C^\infty(\Delta_N)$. We leave this part of the proof to the reader. To investigate the smoothness in x, it is useful to transform (14.1) to get an equivalent family of equations on a space that does not depend on x. See also [21, Section 2.3] for a discussion of these issues.

Once we know that p is smooth, we can get integral equations for the derivates by differentiating (14.1). Since, for the time being, all functions are C^∞, we may differentiate under the integral sign. Thus we obtain

$$p_t(x,t) = -\int_0^x K_t(t,s)p(x,s)\,ds + g'(t), \tag{14.3a}$$

$$p_x(x,t) + \int_0^x K(t,s)p_x(x,s)\,ds = -K(t,x)p(x,x), \tag{14.3b}$$

$$\begin{aligned} p_{xx}(x,t) + \int_0^x K(t,s)p_{xx}(x,s)\,ds = &- K_x(t,x)p(x,x) - K(t,x) \\ &(2p_x(x,s) + p_s(x,s))|_{s=x}, \end{aligned} \tag{14.3c}$$

$$p_{xt}(x,t) = -K_t(t,x)p(x,x) - \int_0^x K_t(t,s)p_x(x,s)\,ds. \tag{14.3d}$$

For general ϕ and g, we approximate ϕ' in $L_1(-2N,2N)$ by odd functions $\phi_n' \in C_0^\infty(-2N,2N)$, and we put $\phi_n(x) = \int_0^x \phi_n'(t)\,dt$. Then $\phi_n \to \phi$ in $C[-2N,2N]$ and $K^{(n)} \to K$ in $C(\Delta_N)$ (we use superscripts here because in a moment we will want to denote partial derivatives by subscripts). Similary, we pick $L_1(0,N)$ approximations $g_n'' \in C_0^\infty(0,N)$ of $g'' \in L_1(0,N)$ and put

$$g_n(x) = g(0) + xg'(0) + \int_0^x g_n''(t)(x-t)\,dt.$$

The integral operators $\mathscr{K}_{\phi_n}^{(x)}$ converge to $\mathscr{K}_{\phi}^{(x)}$ in the operator norm of any of the spaces we have considered above. In particular, $1 + \mathscr{K}_{\phi_n}^{(x)}$ is boundedly invertible for all sufficiently large n. In fact, $\mathscr{K}_{\phi_n}^{(x)}$ converges to $\mathscr{K}_{\phi}^{(x)}$ in the norm of $B(L_1(0,x), C[0,x])$. Moreover, this convergence is uniform in $x \in (0,N]$.

Let $p^{(n)}$ be the solution of (14.1) with K and g replaced by $K^{(n)}$ and g_n, respectively. Then the above remarks together with (14.2) imply that

$$\|p^{(n)}(x,\cdot) - p(x,\cdot)\|_{C[0,x]} \to 0,$$

uniformly in $x \in [0,N]$. (Strictly speaking, one needs a separate argument for the degenerate case $x = 0$, but things are very easy here because $p^{(n)}(0,0) = g_n(0) = g(0) = p(0,0)$.) In other words, $p^{(n)}$ converges to the solution p of the original problem in $C(\Delta_N)$. In particular, $p \in C(\Delta_N)$. Similar arguments work for the first order partial derivatives. Eq. (14.3b) says that

$$p_x^{(n)}(x,\cdot) = -p^{(n)}(x,x)(1 + \mathscr{K}_{\phi_n}^{(x)})^{-1} K^{(n)}(\cdot,x),$$

and the right-hand side converges in $C[0,x]$, uniformly with respect to x. Again, the case $x = 0$ needs to be discussed separately; we leave this to the reader. It follows that the partial derivative p_x exists, is continuous and is equal to this limit function. The argument for the existence and continuity of p_t, which uses (14.3a), is similar (perhaps easier, because one does not need to invert an operator). We have proved part (a) now.

Eq. (14.3c) (for $K^{(n)}$ instead of K) again has the form

$$(1 + \mathscr{K}_{\phi_n}^{(x)})p_{xx}^{(n)}(x,\cdot) = h_n(x,\cdot);$$

we do not write out the inhomogeneous term h_n here. Since h_n converges in $L_1(0,x)$, but not necessarily in $C[0,x]$, we now only obtain convergence of $p_{xx}^{(n)}$ in $L_1(0,x)$. We denote the limit function by p_{xx}, so $p_{xx}(x,\cdot) \in L_1(0,x)$ and

$$\|p_{xx}^{(n)}(x,\cdot) - p_{xx}(x,\cdot)\|_{L_1(0,x)} \to 0,$$

uniformly in x. Note, however, that p_{xx} need not be a partial derivative in the classical sense. Using similar arguments, we deduce from (14.3d) that $p_{xt}^{(n)}(x,\cdot)$ converges in $L_1(0,x)$ to a limit function, which we denote by $p_{xt}(x,\cdot)$. As usual, the convergence is uniform in x.

We have that

$$p'(x,x) = (p_x + p_t)|_{t=x} = -p(x,x)K(x,x) + g'(x) - \int_0^x K(x,s)p_x(x,s)\,ds - \int_0^x K_x(x,s)p(x,s)\,ds. \tag{14.4}$$

We now show that the individual terms on the right-hand side are in $AC^{(1)}[0,N]$. This is obvious for the first two terms, so we only need to discuss the integrals. If we replace K and p in these integrals by $K^{(n)}$ and $p^{(n)}$, respectively, and then let n tend to infinity, we have convergence to the original terms in $C[0,N]$. We can therefore prove absolute continuity of these terms by showing that the derivatives converge in $L_1(0,N)$. So, consider

$$\frac{d}{dx}\int_0^x K^{(n)}(x,s)p_x^{(n)}(x,s)\,ds = K^{(n)}(x,x)p_x^{(n)}(x,x) + \int_0^x K_x^{(n)}(x,s)p_x^{(n)}(x,s)\,ds + \int_0^x K^{(n)}(x,s)p_{xx}^{(n)}(x,s)\,ds.$$

It is easy to see that the first two terms on the right-hand side converge in $C[0,N]$. As for the last term, we note that

$$\int_0^x K^{(n)}(x,s)p_{xx}^{(n)}(x,s)\,ds = \int_0^x K(x,s)p_{xx}^{(n)}(x,s)\,ds + \int_0^x (K^{(n)}(x,s) - K(x,s))p_{xx}^{(n)}(x,s)\,ds.$$

Now recall that $\mathscr{K}_{\phi_n}^{(x)} - \mathscr{K}_{\phi}^{(x)} \to 0$ in $B(L_1, C)$ (uniformly in x) and $\|p_{xx}^{(n)}(x,\cdot)\|_{L_1(0,x)}$ is bounded as a function of n and x. Therefore, the last term goes to zero in $C[0,N]$. Similarly, the first term also converges in $C[0,N]$, as we see from the following estimate:

$$\left|\int_0^x K(x,s)(p_{xx}^{(n)}(x,s) - p_{xx}(x,s))\,ds\right| \leqslant \|\mathscr{K}_{\phi}^{(x)}\|_{B(L_1,C)}\|p_{xx}^{(n)}(x,\cdot) - p_{xx}(x,\cdot)\|_{L_1}.$$

Finally, let us analyze the last term from (14.4). By definition of K (see (4.1)),

$$K_x(x,s) = \tfrac{1}{2}(\phi'(x-s) + \phi'(x+s)).$$

Let us look at the term with $\phi'(x-s)$; the other term is of course treated similarly. An integration by parts gives

$$\int_0^x \phi'(x-s)p(x,s)\,ds = \phi(x)p(x,0) + \int_0^x \phi(x-s)p_s(x,s)\,ds.$$

The first term on the right-hand side manifestly is absolutely continuous. To establish absolute continuity of the integral, we argue exactly as above. Namely, we approximate by smooth functions and compute the derivative:

$$\begin{aligned}&\frac{d}{dx}\int_0^x \phi_n(x-s)p_s^{(n)}(x,s)\,ds\\ &\quad= \int_0^x \phi_n'(x-s)p_s^{(n)}(x,s)\,ds + \int_0^x \phi_n(x-s)p_{xs}^{(n)}(x,s)\,ds.\end{aligned}$$

Now arguments analogous to those used in the preceding section show that this derivative converges in $C[0,N]$, and, also as above, convergence in $L_1(0,N)$ already would have been sufficient to deduce the required absolute continuity. $\square$

15. Some identities

First of all, we can now complete the work of Section 11.

Theorem 15.1. *There exists $H(x)\in L_1(0,N)$, $H(x)\geqslant 0$ for almost every $x\in(0,N)$, $H\not\equiv 0$ on nonempty open sets, so that for all $x\in[0,N]$, we have that $B_x = H_x$ (as de Branges spaces). Here, B_x is the de Branges space based on the de Branges function $E_x(z) = u_1(x,z) + iu_2(x,z)$, where*

$$Ju'(t) = zH(t)u(t), \quad u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

as in Proposition 6.1, and H_x is the space from Lemma 9.1.

Moreover, $H(x)$ can be chosen so that $\hat{F}(0) = \tilde{F}(0)$ for all $F\in B_N = H_N$.

The last part justifies our definition of $K_0^{(x)}\in H_x$ from Section 13. Recall also that $\hat{F}(0)$ is computed in H_N, while $\tilde{F}(0)$ is computed as in Section 12 by using the realization B_N of this space.

Proof. As explained in Section 11, we use the system from Proposition 11.2, but with $x(t)$ as the new independent variable.

As the first step of the proof, let us check how far we are from satisfying the last part of Theorem 15.1. Propositions 12.2 and 13.1 show that

$$\tilde{F}(0)\overline{G(0)} - F(0)\overline{\tilde{G}(0)} = \hat{F}(0)\overline{G(0)} - F(0)\overline{\hat{G}(0)}$$

for all $F, G \in B_N = H_N$. In particular, if $F(0) = 0$, then $\tilde{F}(0) = \hat{F}(0)$. Since both $F \mapsto \tilde{F}(0)$ and $F \mapsto \hat{F}(0)$ are linear maps, it follows that there exists a constant $c \in \mathbb{C}$, independent of F, so that

$$\hat{F}(0) = \tilde{F}(0) + cF(0). \tag{15.1}$$

From the definitions, we see that $\widetilde{F^{\#}}(0) = \overline{\tilde{F}(0)}$ and $\widehat{F^{\#}}(0) = \overline{\hat{F}(0)}$, so the c from (15.1) must actually be real.

To avoid confusion, let us temporarily denote the reproducing and conjugate kernels (for $z = 0$) of the spaces B_t by $j_0^{(t)}$ and $k_0^{(t)}$ (lowercase letters!), respectively. Note also that the conjugate and reproducing kernels depend only on the de Branges space, but not on the particular de Branges function chosen. Therefore (15.1) says that $K_0^{(N)}(z) = k_0^{(N)}(z) + cj_0^{(N)}(z)$. Since $B_t = H_{x(t)}$ and since by Lemma 12.1, $\tilde{F}(0)$ does not depend on the space in which the conjugate function is computed, we also have that $j_0^{(t)}(z) = J_0^{(x(t))}(z)$ and

$$K_0^{(x(t))}(z) = k_0^{(t)}(z) + cj_0^{(t)}(z). \tag{15.2}$$

Next, we claim that the following analog of Lemma 11.1 holds:

$$B_t = \overline{\bigcup_{s<t} B_s} = \bigcap_{s>t} B_s. \tag{15.3}$$

We will only prove (and use) this for canonical systems without singular points, where the proof is very easy. However, a similar but—due to the possible presence of singular points—somewhat more complicated result holds for general canonical systems. If $S = \emptyset$, then Lemma 10.4 implies that $R_{(a,b)} = \{0\}$ for arbitrary $a<b$. Thus the U from (10.4a) and (10.4b) maps $L_2^H(0,t)$ unitarily onto B_t for all $t \in [0,N]$. In particular, the following obvious fact is equivalent to (15.3):

$$L_2^H(0,t) = \overline{\bigcup_{s<t} L_2^H(0,s)} = \bigcap_{s>t} L_2^H(0,s).$$

It also follows that $x(t)$ is strictly increasing. Indeed, if $x(t_1) = x(t_2)$, then $B_{t_1} = B_{t_2}$, and these spaces are mapped by U^{-1} onto $L_2^H(0,t_1)$ and $L_2^H(0,t_2)$, respectively, so $t_1 = t_2$.

Relation (15.3) allows us to show that $x(t)$ is continuous. Proposition 11.2 together with (15.3) implies that

$$B_t = \bigcap_{s>t} B_s = \bigcap_{s>t} H_{x(s)}.$$

On the other hand, $B_t = H_{x(t)}$, and now Lemma 11.1 and the fact that $H_y \ominus H_x$ is not the zero space for $y > x$ show that $\inf_{s>t} x(s) \leqslant x(t)$. A similar argument gives $\sup_{s<t} x(s) \geqslant x(t)$. Since $x(t)$ is monotonically increasing, these two properties suffice to deduce that $x(t)$ is continuous.

Let $t(x)$ be the inverse function of $x(t)$. Then t is also strictly increasing and continuous. We want to show that t is actually absolutely continuous. To this end, we compare $j_0(0) + \widetilde{k_0}(0)$ in the spaces $B_{t(x)} = H_x$. Specializing Proposition 6.1 to $z = \zeta = 0$ (and replacing N by $t(x)$), we see that

$$j_0^{(t(x))}(0) = \int_0^{t(x)} H_{11}(s)\, ds.$$

Moreover, arguing as in the proof of Lemma 12.1, we obtain

$$\begin{aligned}\widetilde{k_0^{(t(x))}}(0) &= [k_0^{(t(x))}, k_0^{(t(x))}]_{B_{t(x)}} = \langle v(\cdot,0), v(\cdot,0)\rangle_{L_2^H(0,t(x))}\\ &= \int_0^{t(x)} H_{22}(s)\, ds.\end{aligned}$$

Hence, still working in $B_{t(x)}$, we have that

$$j_0^{(t(x))}(0) + \widetilde{k_0^{(t(x))}}(0) = \int_0^{t(x)} \operatorname{tr} H(s)\, ds = \tau t(x), \tag{15.4}$$

where τ is a positive constant.

On the other hand, we may use (15.1) and (15.2) to evaluate $j_0(0) + \widetilde{k_0}(0)$ in H_x. We compute

$$\widetilde{k_0^{(t(x))}}(0) = \widetilde{K_0^{(x)}}(0) - c\widetilde{J_0^{(x)}}(0) = \widehat{K_0^{(x)}}(0) - cK_0^{(x)}(0) - c\widehat{J_0^{(x)}}(0) + c^2 J_0^{(x)}(0).$$

By definition of w and the transform $F \mapsto \hat{F}(0)$, we have $\widehat{K_0^{(x)}}(0) = \int_0^x w(x,t)\psi(t)\, dt$, and this is an absolutely continuous function of $x \in [0,N]$ by Theorem 14.1. Similar arguments apply to the other terms and to $j_0^{(t(x))}(0) = J_0^{(x)}(0) = \int_0^x y(x,t)\, dt$. Comparing with (15.4), we thus conclude that $t(x) \in AC^{(1)}[0,N]$, as desired.

We are now ready to transform (11.1). Define $\tilde{u}(x) = u(t(x))$ and $\tilde{H}(x) = t'(x)H(t(x))$. Since $t' \geqslant 0$ almost everywhere, this $\tilde{H}$ is in $L_1(0,N)$ and positive semidefinite almost everywhere. As $t(x)$ is strictly increasing, t' cannot vanish identically on a nonempty open set, and thus $\tilde{H}$ also has this property. Moreover, $\tilde{u}$ solves the corresponding canonical equation:

$$J\tilde{u}'(x) = Jt'(x)u'(t(x)) = zt'(x)H(t(x))u(t(x)) = z\tilde{H}(x)\tilde{u}(x).$$

By definition of $\tilde{u}$, the new de Branges spaces $\tilde{B}_x$ are related to the old spaces by $\tilde{B}_x = B_{t(x)}$. Hence $\tilde{B}_x = H_x$, as desired.

Finally, we can get rid of c in (15.1) by passing to the new matrix

$$H_c(x) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} H(x) \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

(To avoid clumsy notation, we write $H(x)$ instead of $\tilde{H}(x)$ for the H constructed above.) Let $u^{(c)}$, $v^{(c)}$ be the solutions of the transformed system

$$Jy'(x) = zH_c(x)y(x) \tag{15.5}$$

with the initial values $u^{(c)}(0,z) = \binom{1}{0}$, $v^{(c)}(0,z) = \binom{0}{1}$. These functions are related to the old solutions $u = u^{(0)}$ and $v = v^{(0)}$ by

$$u^{(c)}(x,z) = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} u(x,z), \quad v^{(c)}(x,z) = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} (cu(x,z) + v(x,z)).$$

So, first of all, since the de Branges functions associated with (15.5) are given by $E_x^{(c)}(z) = u_1^{(c)}(x,z) + iu_2^{(c)}(x,z)$, it follows from Theorem 7.2 that (15.5) still generates the same de Branges spaces: $B_x \equiv B(E_x^{(0)}) = B(E_x^{(c)})$. So the equality $B_x = H_x$ continues to hold if we replace H by H_c. Second, a calculation based on (12.1) shows that the conjugate kernel $K_z^{(c)}$ of $B(E_N^{(c)})$ (I apologize for the slightly inconsistent notation, but there are so many conjugate kernels in this argument) is given by $K_z^{(c)}(\zeta) = K_z(\zeta) + cJ_z(\zeta)$, and thus for the new system (15.5), we have that $\tilde{F}(0) = \hat{F}(0)$. $\square$

Proposition 15.2. *Let H and y, w be as in Theorems* 15.1 *and* 13.2, *respectively. Then*

$$H_{11}(x) = y(x,x) + \int_0^x y_x(x,t)\,dt,$$

$$H_{12}(x) = w(x,x) + \int_0^x w_x(x,t)\,dt = y(x,x)\psi(x) + \int_0^x y_x(x,t)\psi(t)\,dt,$$

$$H_{22}(x) = w(x,x)\psi(x) + \int_0^x w_x(x,t)\psi(t)\,dt.$$

Proof. Basically, we have established this already in the preceding proof when we showed that $t(x)$ is absolutely continuous. The idea is to compare reproducing and conjugate kernels in $B_x = H_x$.

By Proposition 6.1, the reproducing kernel $J_0^{(x)}$ of B_x, evaluated at $z = 0$, is given by $J_0^{(x)}(0) = \int_0^x H_{11}(t)\,dt$. On the other hand, in H_x we have $J_0^{(x)}(0) = \int_0^x y(x,t)\,dt$, so

$$\int_0^x H_{11}(t)\,dt = \int_0^x y(x,t)\,dt.$$

Now take the derivatives with respect to x, using Theorem 14.1 on the right-hand side. The formula for H_{11} follows.

Similarly, (12.2) shows that in B_x, we have $K_0^{(x)}(0) = \int_0^x H_{12}(t)\,dt$, and now the same reasoning applies and gives the first formula for H_{12}. To prove the second one, notice that the conjugate of $J_0^{(x)}$, evaluated in H_x, is given by

$$\widehat{J_0^{(x)}}(0) = \int_0^x y(x,t)\psi(t)\,dt.$$

On the other hand, $\widetilde{J_0^{(x)}}(0) = [K_0^{(x)}, J_0^{(x)}]_{B_x}$ by Lemma 12.1. This scalar product can be evaluated with the help of the U from (10.4b), because $K_0^{(x)} = U(\chi_{(0,x)}v(\cdot,0))$ by (12.2) and $J_0^{(x)} = U(\chi_{(0,x)}u(\cdot,0))$ by Proposition 6.1 (and a simple manipulation). Thus

$$\widetilde{J_0^{(x)}}(0) = \langle v(\cdot,0), u(\cdot,0)\rangle_{L_2^H(0,x)} = \int_0^x H_{21}(t)\,dt = \int_0^x H_{12}(t)\,dt,$$

and the second formula for H_{12} now follows.

Finally, the formula for H_{22} is proved by an analogous argument, using the conjugate of $K_0^{(x)}$ this time. □

Our final goal is to verify that $H(x)$ satisfies the hypotheses of Proposition 8.1. Proposition 15.2 gives some hope that this can be done by analyzing the functions y, w from Theorem 13.2.

Proposition 15.3. *Let y, w be as in Theorem* 13.2. *Then* $y(0,0) = 1$, $w(0,0) = 0$, $y'(0,0) = 0$, $w'(0,0) = 1$, *and*

$$y(x,x)w'(x,x) - y'(x,x)w(x,x) = 1.$$

Proof. By the usual approximation argument, it suffices to prove this under the additional assumption that $\phi \in C^\infty$. Consider again the general version (14.1) of the integral equations for y and w. By differentiating with respect to x, we see that p_x satisfies

$$(1 + \mathscr{K}_\phi^{(x)})p_x(x,\cdot) = -p(x,x)K(\cdot,x). \tag{15.6}$$

Also, p_{tt} solves

$$p_{tt}(x,t) + \int_0^x K_{tt}(t,s)p(x,s)\,ds = g''(t). \tag{15.7}$$

Since $K_{tt}(t,s) = K_{ss}(t,s)$, we can use integration by parts to rewrite this equation. In the following calculation, we will use the notation ∂_i for the partial derivative with respect to the ith variable ($i = 1,2$).

$$\begin{aligned}\int_0^x K_{ss}(t,s)p(x,s)\,ds &= K_x(t,x)p(x,x) - \partial_2 K(t,0)p(x,0)\\ &\quad - \int_0^x K_s(t,s)p_s(x,s)\,ds\\ &= K_x(t,x)p(x,x) - K(t,x)\partial_2 p(x,x) + \phi(t)\partial_2 p(x,0)\\ &\quad + \int_0^x K(t,s)p_{ss}(x,s)\,ds.\end{aligned}$$

We have used the fact that because ϕ is even, $\partial_2 K(t,0)=0$. Also, $K(t,0)=\phi(t)$. Plug this into (15.7) and subtract the resulting equation from Eq. (14.3c) for p_{xx}. There are some cancellations, and the function $P=p_{xx}-p_{tt}$ solves the relatively simple equation

$$(1+\mathcal{K}_\phi^{(x)})P(x,\cdot)=-2p'(x,x)K(\cdot,x)+\partial_2 p(x,0)\phi-g''. \tag{15.8}$$

By putting $t=0$ in Eq. (14.3a) for p_t and noting that $\partial_1 K(0,s)=0$, we see that $\partial_2 p(x,0)=g'(0)$. Now if g is one of the functions from the equations of Theorem 13.2 (so $g(t)=1$ or $g(t)=\psi(t)$), then $g'(0)\phi(t)-g''(t)\equiv 0$. Hence (15.8) says that the functions $Y=y_{xx}-y_{tt}$ and $W=w_{xx}-w_{tt}$ solve

$$(1+\mathcal{K}_\phi^{(x)})Y(x,\cdot)=-2y'(x,x)K(\cdot,x),$$

$$(1+\mathcal{K}_\phi^{(x)})W(x,\cdot)=-2w'(x,x)K(\cdot,x).$$

Comparison with (15.6) for $p=y$ and w shows that

$$\frac{Y(x,t)}{2y'(x,x)}=\frac{W(x,t)}{2w'(x,x)}=\frac{y_x(x,t)}{y(x,x)}=\frac{w_x(x,t)}{w(x,x)}\quad (0\leqslant t\leqslant x\leqslant N). \tag{15.9}$$

More precisely, the ratios whose denominators are different from zero are equal to one another; if a denominator equals zero, then the corresponding numerator is identically equal to zero for $t\in[0,x]$. (As usual, the case $x=0$ should be discussed separately, but, also as usual, we leave this to the reader.) It follows from (15.9) that

$$y_x(x,t)w(x,x)=w_x(x,t)y(x,x)\quad (0\leqslant t\leqslant x\leqslant N), \tag{15.10}$$

and this holds in all cases. Take derivatives with respect to x,

$$y_{xx}(x,t)w(x,x)+y_x(x,t)w'(x,x)=w_{xx}(x,t)y(x,x)+w_x(x,t)y'(x,x),$$

and subtract twice this equation from the identity

$$2w'(x,x)y_x(x,t)=(w_{xx}(x,t)-w_{tt}(x,t))y(x,x),$$

which follows from (15.9). We obtain

$$\begin{aligned}(w_{xx}(x,t)&+w_{tt}(x,t))y(x,x)\\ &=-2w_x(x,t)y'(x,x)+2w(x,x)y_{xx}(x,t).\end{aligned} \tag{15.11}$$

If we interchange the roles of y and w, we get the analogous identity

$$\begin{aligned}(y_{xx}(x,t)&+y_{tt}(x,t))w(x,x)\\ &=-2y_x(x,t)w'(x,x)+2y(x,x)w_{xx}(x,t).\end{aligned} \tag{15.12}$$

Now subtract (15.11) from (15.12). On the right-hand side, we get zero because

$$\begin{aligned}&-2y_x(x,t)w'(x,x)+2y(x,x)w_{xx}(x,t)+2w_x(x,t)y'(x,x)\\ &\quad-2w(x,x)y_{xx}(x,t)=2\partial_x(y(x,x)w_x(x,t)-w(x,x)y_x(x,t)),\end{aligned}$$

and the expression in parentheses is zero by (15.10). Hence

$$(w_{xx}(x,t)+w_{tt}(x,t))y(x,x)=(y_{xx}(x,t)+y_{tt}(x,t))w(x,x).$$

Since $y_{tx}(x,t)w(x,x) = w_{tx}(x,t)y(x,x)$ by (15.10) again, we also have that

$$\begin{aligned}(w_{xx}(x,t) &+ 2w_{tx}(x,t) + w_{tt}(x,t))y(x,x)\\ &= (y_{xx}(x,t) + 2y_{tx}(x,t) + y_{tt}(x,t))w(x,x). \end{aligned} \tag{15.13}$$

By the chain rule,

$$w''(x,x) = (w_{xx}(x,t) + 2w_{tx}(x,t) + w_{tt}(x,t))|_{t=x},$$

so taking $t = x$ in (15.13) yields

$$\begin{aligned}w(x,x)&y''(x,x) - y(x,x)w''(x,x)\\ &= \frac{d}{dx}(w(x,x)y'(x,x) - w'(x,x)y(x,x)) = 0.\end{aligned}$$

We determine the constant value of $wy' - w'y$ by evaluating at $x = 0$. Since $K(0,0) = 0$, we see directly from integral equation (14.1) that $p(x,x) = g(x) + O(x^2)$ for small $x > 0$, so $p(0,0) = g(0)$, $p'(0,0) = g'(0)$. Hence

$$y(0,0) = 1, \quad w(0,0) = 0, \quad y'(0,0) = 0, \quad w'(0,0) = 1,$$

as claimed, and it also follows that $yw' - y'w = 1$. $\square$

Proposition 15.4. *Let H and y, w be as in Theorems* 15.1 *and* 13.2, *respectively. Then*

$$H_{11}(x)w(x,x) = H_{12}(x)y(x,x), \quad H_{12}(x)w(x,x) = H_{22}(x)y(x,x).$$

Proof. These identities follow at once from Proposition 15.1 and (15.10). $\square$

16. Conclusion of the proof of Theorem 5.1

We are now in a position to verify the hypotheses of Proposition 8.1 for the H constructed in Theorem 15.1. More precisely, we will transform the canonical system one more time to obtain a new system satisfying the assumptions of Proposition 8.1. At the end, however, it will turn out that this transformation was actually unnecessary.

We know from Theorem 14.1(b) that the functions $y(x,x)$, $w(x,x)$ belong to $AC^{(2)}[0,N]$, and Proposition 15.3 implies that if $y(x_0,x_0) = 0$, then $y'(x_0,x_0) \neq 0$, $w(x_0,x_0) \neq 0$. Thus y, w have only finitely many zeros in $[0,N]$ and they do not vanish simultaneously. Also, Proposition 15.4 implies that $H_{11}w^2 = H_{22}y^2$. So we may consistently define a function $r \geqslant 0$ by

$$r(x) = \begin{cases} |y(x,x)|^{-1}\sqrt{H_{11}(x)} & \text{if } y(x,x) \neq 0,\\ |w(x,x)|^{-1}\sqrt{H_{22}(x)} & \text{if } w(x,x) \neq 0.\end{cases}$$

We now need a certain regularity of r (more precisely, we need that $r \in AC^{(2)}$). This can be established directly by showing that $H_{ij} \in AC^{(2)}[0, N]$. Note, however, that this statement is not obvious at this point because, for example, the second derivative H_{11}'', evaluated formally with the help of Proposition 15.2, contains the third order derivative y_{xxx}, which need not exist. Thus it is again easier to first carry out this final part of the proof of Theorem 5.1 under the additional assumption that $\phi \in C^\infty$ and then pass to the general case by a limiting argument.

By Propositions 15.2 and 15.3, $y(0,0) = H_{11}(0) = 1$, hence $r(0) = 1$. Also, since we are assuming that $\phi \in C^\infty$, the function r is also smooth as long as $r > 0$. Fix an interval $[0, L] \subset [0, N]$, so that $r > 0$ on $[0, L]$. On this interval $[0, L]$, we transform the canonical system as follows. Let

$$t(x) = \int_0^x r(s)\, ds \quad (0 \leqslant x \leqslant L),$$

let $x(t)$ be the inverse function, and define the new matrix $\tilde{H}(t) = H(x(t))/r(x(t))$ for $0 \leqslant t \leqslant t(L)$. Let $u(x, z)$ be the solution of the original system

$$Ju' = zHu, \quad u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and put $\tilde{u}(t, z) = u(x(t), z)$. Then $\tilde{u}$ solves the new equation

$$J\tilde{u}' = z\tilde{H}\tilde{u}, \quad \tilde{u}(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \tag{16.1}$$

the corresponding de Branges spaces are related by $\tilde{B}_t = B_{x(t)}$.

Now the first line from the definition of r shows that

$$\tilde{H}_{11}(t) = r(x(t)) \frac{H_{11}(x(t))}{r^2(x(t))} = r(x(t)) y^2(x(t)), \tag{16.2}$$

at least if $y(x(t)) \neq 0$. Here, $y(x)$ is short-hand for $y(x, x)$. However, as y and w do not vanish simultaneously, Proposition 15.4 implies that $H_{11}(x) = 0$ if $y(x) = 0$, so (16.2) holds generally. Similar arguments apply to the other matrix elements:

$$\tilde{H}(t) = r(x(t)) \begin{pmatrix} y^2(x(t)) & (yw)(x(t)) \\ (yw)(x(t)) & w^2(x(t)) \end{pmatrix} \equiv \begin{pmatrix} a^2(t) & (ab)(t) \\ (ab)(t) & b^2(t) \end{pmatrix},$$

where $a(t) = r^{1/2}(x(t))y(x(t))$, $b(t) = r^{1/2}(x(t))w(x(t))$. Now as $r > 0$ on $[0, L]$, we have $a, b \in C^{\infty}[0, t(L)]$ and

$$\begin{aligned}
&a(t)b'(t) - a'(t)b(t) \\
&\quad = r^{1/2}(x(t))\left(y(x(t))\frac{d}{dt}(r^{1/2}(x(t))w(x(t))) \right. \\
&\qquad \left. - w(x(t))\frac{d}{dt}(r^{1/2}(x(t))y(x(t)))\right) \\
&\quad = r(x(t))\left(y(x(t))\frac{d}{dt}w(x(t)) - w(x(t))\frac{d}{dt}y(x(t))\right) \\
&\quad = (y(x)w'(x) - w(x)y'(x))|_{x=x(t)} = 1
\end{aligned}$$

by Proposition 15.3. Moreover, $a(0) = r^{1/2}(0)y(0) = 1$ and also by Proposition 15.3 and the fact that $\partial_1 y(0,0) = 0$. Thus canonical system (16.1) satisfies the assumptions of Proposition 8.1. So (16.1) comes from a Schrödinger equation. In particular, we have the following description of $\widetilde{B}_t$ as a set:

$$\tilde{B}_t = S_t = \left\{ F(z) = \int_0^t f(s)\cos\sqrt{z}s\, ds \colon f \in L_2(0,t)\right\}.$$

On the other hand, $\tilde{B}_t = B_{x(t)} = H_{x(t)}$ by Theorem 15.1, and, again as sets, $H_{x(t)} = S_{x(t)}$ by the definition of $H_{x(t)}$. We are forced to admit that $x(t) = t$ for all $t \in [0, t(L)]$.

In other words, we have shown that if $r > 0$ on $[0, L]$, then $r \equiv 1$ on $[0, L]$. Also, as noted at the beginning of the argument, $r(0) = 1$, so the set of L's such that $r \equiv 1$ on $[0, L]$ is nonempty and closed and open in $[0, N]$, hence $r \equiv 1$ on all of $[0, N]$.

So in reality, there has been no transformation, and system (16.1) is the system from Theorem 15.1. This system is equivalent to a Schrödinger equation, that is, there exists $V \in L_1(0, N)$, so that $H_x = B_x = S_x$ (as de Branges spaces). In particular, we may specialize to $x = N$, and we have thus proved Theorem 5.1 under the additional assumption that $\phi \in C^{\infty}$.

The extension to the general case is routine. As usual, approximate ϕ' in $L_1(-2N, 2N)$ by odd functions $\phi_n' \in C_0^{\infty}(-2N, 2N)$ and put $\phi_n(x) = \int_0^x \phi_n'(t)\, dt$. Then $\phi_n \in C^{\infty} \cap \Phi_N$ for all sufficiently large n.

As a by-product of the above argument, we have the formulae

$$\begin{aligned}
&H_{11}(x) = y^2(x,x), \quad H_{12}(x) = y(x,x)w(x,x), \\
&\quad H_{22}(x) = w^2(x,x),
\end{aligned} \tag{16.3}$$

which are valid for smooth ϕ. So we may use (16.3) if we replace ϕ by ϕ_n. Now if $n \to \infty$, all quantities converge pointwise to the right limits; for the matrix elements H_{ij}, this follows from Proposition 15.2. So (16.3) holds in the general case as well. Now a glance at Theorem 14.1(b) and Proposition 15.3 suffices to verify the hypotheses of Proposition 8.1 (for the canonical system from Theorem 5.1; no transformation is needed this time). This completes the proof of Theorem 5.1.

17. Half line problems

In this section, we discuss half line problems, that is, operators of the form $-d^2/dx^2 + V(x)$ on $L_2(0, \infty)$. We assume, as usual, that $V \in L_{1,\mathrm{loc}}([0, \infty))$. Our presentation in this section will be less detailed.

Of course, in a sense, half line problems are contained in our previous treatment because we may analyze the problem on $(0, \infty)$ by analyzing it on $(0, N)$ for every N. More precisely, Theorems 4.1, 4.2, 5.1 and 5.2, applied with variable $N > 0$, give a one-to-one correspondence between functions $\phi \in \bigcap_{N>0} \Phi_N$ and locally integrable potentials $V : [0, \infty) \to \mathbb{R}$. Here we say that $\phi \in \bigcap_{N>0} \Phi_N$ if the restriction of ϕ to $[-2N, 2N]$ belongs to Φ_N for every $N > 0$. The uniqueness assertions from Theorem 5.2 make sure that there are no consistency problems. For example, the following holds: If $N_1 < N_2$, $\phi_{N_i} \in \Phi_{N_i}$ and $\phi_{N_1} = \phi_{N_2}$ on $[-2N_1, 2N_1]$, then, by Theorem 5.2(b), the corresponding potentials satisfy $V_1 = V_2$ on $(0, N_1)$.

This local treatment looks most natural and satisfactory, but it is also reasonable to ask for conditions that characterize the spectral measures of half line problems. In particular, this will relate our results to the Gelfand–Levitan characterization of spectral data.

Given a potential $V \in L_{1,\mathrm{loc}}([0, \infty))$, we call a positive Borel measure ρ on $\mathbb{R}$ a *spectral measure* of the half line problem if the de Branges spaces S_N are isometrically contained in $L_2(\mathbb{R}, d\rho)$ for all $N > 0$. In other words, we demand that

$$\|F\|_{S_N}^2 = \int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho(\lambda) \quad \forall F \in \bigcup_{N>0} S_N.$$

Borrowing the terms commonly used for discrete problems, we may also say that the spectral measures are precisely the solutions of a (continuous version of a) certain moment problem. By Theorem 3.2(b), the measures from Weyl theory are indeed spectral measures in this sense. In particular, given a potential $V \in L_{1,\mathrm{loc}}([0, \infty))$, spectral measures always exist. The spectral measure is unique precisely if V is in the limit point case at infinity. Indeed, if V is in the limit circle case, any choice of a boundary condition at infinity yields a spectral measure, and there are many others. For instance, one can form convex combinations or, more generally, averages of these measures. Conversely, if V is in the limit point case, then uniqueness of the spectral measure follows from the Nevanlinna type parametrization of the measures μ for which $L_2(\mathbb{R}, d\mu)$ isometrically contains S_N together with the fact that the Weyl circles shrink to points.

The Gelfand–Levitan conditions characterize the spectral measures of half line problems. We now want to demonstrate that such a characterization also follows in a rather straightforward way from our direct and inverse spectral theorems (Theorems 4.1, 4.2, 5.1 and 5.2) and some standard material.

For a positive Borel measure ρ, introduce the signed measure $\sigma = \rho - \rho_0$ (where ρ_0 is the measure for zero potential from (4.9)), and consider the following two conditions:

(1) If $F \in \bigcup_{N>0} S_N$, $\int |F(\lambda)|^2 \, d\rho(\lambda) = 0$, then $F \equiv 0$.

(2) For every $g \in C_0^\infty(\mathbb{R})$, the integral $\int d\sigma(\lambda) \int dx\, g(x) \cos\sqrt{\lambda}x$ converges absolutely:

$$\int_{-\infty}^{\infty} d|\sigma|(\lambda) \left| \int_{-\infty}^{\infty} dx\, g(x) \cos\sqrt{\lambda}x \right| < \infty.$$

Moreover, there exists an even, real valued function $\phi \in AC^{(1)}(\mathbb{R})$ with $\phi(0) = 0$, so that

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \int_{-\infty}^{\infty} dx\, g(x) \cos\sqrt{\lambda}x = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx$$

for all $g \in C_0^\infty(\mathbb{R})$.

The set of ρ's satisfying these two conditions will be denoted by GL, for Gelfand–Levitan. We do *not* require that $\bigcup S_N \subset L_2(\mathbb{R}, d\rho)$, so at this point, we cannot exclude the possibility that for fixed $\rho \in GL$, there exists $F \in \bigcup S_N$ with $\int |F|^2 \, d\rho = \infty$. However, we will see in a moment that actually there are no such F's.

Our definition of GL is inspired by Marchenko's treatment of the Gelfand–Levitan theory (see especially [23, Theorem 2.3.1]). Note, however, that Marchenko does not regularize by subtracting ρ_0, but by using the analog of the function ψ from Section 13 instead of ϕ. Moreover, he uses a space of test functions tailor made for the discussion of Schrödinger operators, and he assumes continuity of the potential.

Theorem 17.1. (a) *For every* $\rho \in GL$, *there exists a unique* $V \in L_{1,\mathrm{loc}}([0, \infty))$ *so that* ρ *is a spectral measure of* $-d^2/dx^2 + V(x)$.

(b) *If* ρ *is a spectral measure of* $-d^2/dx^2 + V(x)$, *then* $\rho \in GL$.

Proof. (a) A computation using condition (2) from the definition of GL shows that for every $f \in C_0^\infty(\mathbb{R})$, the function

$$F(\lambda) = \int_{\mathbb{R}} f(x) \cos\sqrt{\lambda}x \, dx$$

belongs to $L_2(\mathbb{R}, d\rho)$ and

$$\|F\|^2_{L_2(\mathbb{R},d\rho)} = \|f\|^2_{L_2(\mathbb{R})} + \operatorname{Re}\int_{-\infty}^{\infty} \overline{f(-t)}f(t)\,dt + \int_{\mathbb{R}}\int_{\mathbb{R}} ds\,dt\,\overline{f(s)}f(t)\frac{1}{2}(\phi(s-t)+\phi(s+t)).$$

The first two terms come from the Plancherel type relation

$$\int_{\mathbb{R}} |F(\lambda)|^2\,d\rho_0(\lambda) = \|f\|^2_{L_2(\mathbb{R})} + \operatorname{Re}\int_{-\infty}^{\infty} \overline{f(-t)}f(t)\,dt.$$

It follows that the identity

$$\|F\|^2_{L_2(\mathbb{R},d\rho)} = \langle f, (1+\mathscr{K}_\phi)f\rangle_{L_2(0,N)} \tag{17.1}$$

holds if $f \in C_0^\infty(0,N)$. By a density argument and the fact that norm convergent sequences have subsequences that converge almost everywhere, condition (1) now implies that $1+\mathscr{K}_\phi > 0$ as an operator on $L_2(0,N)$. So $\phi \in \Phi_N$, and from Theorem 5.1, we thus get $V \in L_1(0,N)$, so that

$$\|F\|^2_{S_N} = \langle f, (1+\mathscr{K}_\phi)f\rangle_{L_2(0,N)}$$

for all $F \in S_N$. Hence $\|F\|_{S_N} = \|F\|_{L_2(\mathbb{R},d\rho)}$ for all F as above with $f \in C_0^\infty(0,N)$. Again by a density argument, this relation actually holds on all of S_N.

The whole argument works for arbitrary N, and, as observed above, Theorem 5.2(b) implies that there are no consistency problems. We obtain a locally integrable potential V on $[0,\infty)$, so that $\|F\|_{S_N} = \|F\|_{L_2(\mathbb{R},d\rho)}$ for all $F \in \bigcup S_N$. In other words, ρ is a spectral measure of $-d^2/dx^2 + V(x)$.

Uniqueness of V is clear because (17.1) forces us to take V on $(0,N)$ so that the norm on S_N is the one determined by $\mathscr{K}_\phi$; so once ϕ is given, there is no choice by Theorem 5.2(b) again. But clearly ϕ is uniquely determined by the measure σ and hence also by ρ.

(b) Property (1) is obvious from the equality $\|F\|_{L_2(\mathbb{R},d\rho)} = \|F\|_{S_N}$. To establish property (2), we use the well-known estimates ([23, Section 2.4]; compare also [15])

$$\lim_{L\to\infty} \rho((-\infty,-L))e^{a\sqrt{L}} = 0 \quad \forall a>0, \qquad \int_{\mathbb{R}} \frac{d\rho(\lambda)}{1+\lambda^2} < \infty.$$

As $|\sigma| \leqslant \rho + \rho_0$, the absolute convergence of $\int d\sigma(\lambda)\int dx\, g(x)\cos\sqrt{\lambda}x$ for $g \in C_0^\infty$ follows. Moreover, this integral depends continuously on $g \in \mathscr{D} = C_0^\infty(\mathbb{R})$ and hence defines a distribution. Now let $f_1, f_2 \in C_0^\infty(\mathbb{R})$ be even functions. Then

$$F_i(z) \equiv \int_{-\infty}^{\infty} f_i(x)\cos\sqrt{z}x\,dx = 2\int_0^\infty f_i(x)\cos\sqrt{z}x\,dx \in \bigcup S_N,$$

and by a calculation,

$$\begin{aligned}[F_1,F_2]_{S_N} &= \langle F_1,F_2\rangle_{L_2(\mathbb{R},d\rho)}\\ &= 4\langle f_1,f_2\rangle_{L_2(0,\infty)} + \int_{-\infty}^{\infty} d\sigma(\lambda)\int_{-\infty}^{\infty} dx\, g(x)\cos\sqrt{\lambda}x,\end{aligned}$$

where N must be chosen so large that $F_1,F_2\in S_N$ and

$$g(x)\equiv\frac{1}{2}\int_{-\infty}^{\infty}\overline{f_1\left(\frac{x+y}{2}\right)}f_2\left(\frac{x-y}{2}\right)dy. \tag{17.2}$$

On the other hand, we have that

$$[F_1,F_2]_{S_N} = 4\langle f_1,(1+\mathscr{K}_\phi)f_2\rangle = 4\langle f_1,f_2\rangle_{L_2(0,\infty)} + \int_{-\infty}^{\infty} g(x)\phi(x)\,dx,$$

where $\phi\in\bigcap\Phi_N$ is the function from Theorem 4.2. Hence

$$\int_{-\infty}^{\infty} d\sigma(\lambda)\int_{-\infty}^{\infty} dx\, g(x)\cos\sqrt{\lambda}x = \int_{-\infty}^{\infty} g(x)\phi(x)\,dx \tag{17.3}$$

for every g that is of form (17.2) with even $f_i\in C_0^\infty(\mathbb{R})$. We claim that this set of g's is rich enough to guarantee the validity of (17.3) for arbitrary $g\in C_0^\infty(\mathbb{R})$. To see this, one can proceed as follows. By a change of variables, (17.2) becomes

$$g(x)=\int_{-\infty}^{\infty}\overline{f_1(x-u)}f_2(u)\,du = (\overline{f_1}*f_2)(x).$$

We can take $\overline{f_1}$ as an approximate identity, that is, $\overline{f_1(x)}=n\varphi(nx)$, where $\int\varphi=1$ and let $n\to\infty$. It follows that the set of g's of form (17.2) is dense (in the topology of $\mathscr{D}=C_0^\infty(\mathbb{R})$) in the set of even test functions. Moreover, for odd test functions g,

$$\int_{-\infty}^{\infty} d\sigma(\lambda)\int_{-\infty}^{\infty} dx\, g(x)\cos\sqrt{\lambda}x = \int_{-\infty}^{\infty} g(x)\phi(x)\,dx = 0.$$

By combining these facts, we deduce that (17.3) holds for every $g\in C_0^\infty(\mathbb{R})$, as claimed. $\square$

We have no uniqueness statement in Theorem 17.1(b): for a given V, there may be many ρ's. However, this only comes from the fact that we have insisted on working with spectral measures. Clearly, in addition to the bijection $V\leftrightarrow\phi$ between $L_{1,\text{loc}}$ and $\bigcap\Phi_N$ discussed at the beginning of this section, we also have a one-to-one correspondence between potentials V and, let us say, distributions

$$g\mapsto\int_{-\infty}^{\infty} d\sigma(\lambda)\int_{-\infty}^{\infty} dx\, g(x)\cos\sqrt{\lambda}x.$$

However, this distribution determines the measure σ (and thus ρ) only if (in fact, precisely if) we have limit point case at infinity. This remark again confirms our claim that in inverse spectral theory, the function ϕ is the more natural object.

18. Some remarks

The proof of Theorem 5.1 has indicated at least two methods of reconstructing the potential V from the spectral data ϕ. One consists of solving the integral equation for y (say),

$$y(x,t) + \int_0^x K(t,s)y(x,s)\,ds = 1.$$

By (16.3) and Proposition 8.1, $V = y''w' - w''y'$, and since $yw'' = wy''$ and $yw' - y'w = 1$, we can compute the potential V from this solution y by $V(x) = y''(x,x)/y(x,x)$. This way of finding V is quite similar to the Gelfand–Levitan procedure, where one solves the integral equation

$$z(x,t) + \int_0^x K(t,s)z(x,s)\,ds = -K(x,t)$$

for z and computes the potential as $V(x) = z'(x,x)$ (see [21, Chapter 2]). Loosely speaking, our function $y(x,t)$ is a two-point version of the solution $y(x)$ to $-y'' + Vy = 0$ with the initial values $y(0) = 1$, $y'(0) = 0$.

Our proof of Theorem 5.1 also admits a second, completely different interpretation. Namely, the integral equations for y, w may be viewed as an auxiliary tool needed to show that the canonical system that was constructed with the aid of Theorem 7.3 is equivalent to a Schrödinger equation. In other words, if one has a constructive proof of Theorem 7.3, one may apply the corresponding reconstruction procedure and one automatically obtains a canonical system that satisfies the hypotheses of Proposition 8.1, possibly after some modifications: deletion of an initial singular interval, introduction of a new independent variable to match the de Branges spaces and finally a transformation of the type $H \to H_c$, as in the proof of Theorem 15.1. (Actually, this last transformation does not affect $H_{11}(x)$ and, by the above, is thus not needed to compute $V(x)$.) Put differently, this means that work on constructive inverse spectral theory of canonical systems always has implications in the inverse spectral theory of Schrödinger operators as well.

In [6], Theorem 7.3 is proved as follows. The first step is to approximate the de Branges function E by polynomial de Branges functions E_n. The construction of (discrete) canonical systems for E_n can be carried out using elementary methods only (for instance, orthogonalization of polynomials). Finally, one passes to the limit $n \to \infty$. See also [26,32] for completely different views on Theorem 7.3.

As a final remark, we would like to point out that the transformation from a Schrödinger equation to a canonical system regularizes the coefficients. Indeed, $H \in AC^{(2)}$, while in general, one only has $V \in L_1$. This effect will be particularly convenient if one considers Schrödinger operators with, let us say, measures or even more singular distributions as potentials. The theory of canonical systems and de Branges spaces seems to provide us with a particularly appropriate approach to the direct and inverse spectral theory of such operators.

19. Dirichlet boundary conditions

We now consider Schrödinger equation (3.1) with Dirichlet boundary condition at the origin: $y(0) = 0$. In inverse spectral theory, the case of Dirichlet boundary conditions often poses additional technical problems (for instance, in [14,22,23], Dirichlet boundary conditions are not discussed). This seems to hold to a lesser extent for the approach developed in this paper. All results presented so far have direct analogs, and in most cases, no new ideas are needed.

We will now give a very sketchy exposition of these results. In fact, I have already used part of this material in [25]. This reference also contains additional hints concerning the proofs. We continue to use the symbols S_N, $\mathscr{K}_\phi$, ϕ etc., but of course these quantities will not be identical to their counterparts for Neumann boundary conditions.

One now defines $u(x, z)$ as the solution of (3.1) with the initial values $u(0, z) = 0$, $u'(0, z) = 1$. Then, as in Section 3, one can form the de Branges function $E_N(z) = u(N, z) + iu'(N, z)$, and $B(E_N) \equiv S_N$ can again be identified with the spaces $L_2(\mathbb{R}, d\rho_N^\beta)$ from the spectral representation. Theorems 4.1 and 4.2 have direct analogs. More precisely, one always has (that is, independently of the potential $V \in L_1(0, N)$)

$$S_N = \left\{ F(z) = \int_0^N f(x) \frac{\sin \sqrt{z} x}{\sqrt{z}} \, dx \colon f \in L_2(0, N) \right\}.$$

Moreover, for any given $V \in L_1(0, N)$, there exists a real valued, even function $\phi \in AC^{(1)}[-2N, 2N]$ with $\phi(0) = 0$ so that for all $F \in S_N$,

$$\|F\|_{S_N}^2 = \langle f, (1 + \mathscr{K}_\phi) f \rangle_{L_2(0,N)}.$$

Here, $\mathscr{K}_\phi$ again is an integral operator on $L_2(0, N)$, but this time with kernel

$$K(x, t) = \tfrac{1}{2}(\phi(x - t) - \phi(x + t))$$

(note the minus sign!). Finally, we still have the inverse and uniqueness results from Section 5. In other words, there is a one-to-one correspondence between potentials $V \in L_1(0,N)$ and ϕ functions $\phi \in \Phi_N$. Here, we again define Φ_N by

$$\Phi_N = \{\phi \in AC^{(1)}[-2N,2N]\colon \phi \text{ real valued,} \\ \text{even, } \phi(0)=0,\ 1+\mathcal{K}_\phi > 0\}.$$

More on these results can be found in [25]. Note also that the condition that $\phi(0)=0$ now has a somewhat different meaning: in fact, it is a normalization rather than a condition because obviously one can add constants to ϕ without changing $\mathcal{K}_\phi$.

We conclude this paper with a characterization of the half line spectral data in the case of Dirichlet boundary conditions. This result is the analog of Theorem 17.1. The referee has pointed out that this seems to be the first such characterization in this generality ([21], for instance, has continuity assumptions on the potential). We again define spectral measures as those positive Borel measures ρ on $\mathbb{R}$ which integrate functions from $\bigcup S_N$ correctly: $\|F\|_{L_2(\mathbb{R},d\rho)} = \|F\|_{S_N}$ for all $F \in \bigcup S_N$. Also, given ρ, we again introduce the signed measure $\sigma = \rho - \rho_0$, where $d\rho_0(\lambda) = \chi_{(0,\infty)}(\lambda)\frac{\sqrt{\lambda}\,d\lambda}{\pi}$ is the (unique) spectral measure of the half line problem for zero potential. Here are the conditions that characterize spectral measures.

(1) If $F \in \bigcup_{N>0} S_N$, $\int |F(\lambda)|^2\, d\rho(\lambda) = 0$, then $F \equiv 0$.

(2) For every $g \in C_0^\infty(\mathbb{R})$ with $\int g = 0$, the integral $\int d\sigma(\lambda) \int dx\, g(x)\frac{\cos\sqrt{\lambda}x - 1}{\lambda}$ converges absolutely:

$$\int_{-\infty}^{\infty} d|\sigma|(\lambda) \left| \int_{-\infty}^{\infty} dx\, g(x)\frac{\cos\sqrt{\lambda}x - 1}{\lambda} \right| < \infty.$$

Moreover, there exists an even, real valued function $\phi \in AC^{(1)}(\mathbb{R})$ with $\phi(0)=0$, so that

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \int_{-\infty}^{\infty} dx\, g(x)\frac{\cos\sqrt{\lambda}x - 1}{\lambda} = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx$$

for all $g \in C_0^\infty(\mathbb{R})$ with $\int g = 0$.

Here, we of course interpret $\frac{\cos\sqrt{\lambda}x-1}{\lambda}\big|_{\lambda=0} = -x^2/2$. For $\lambda \neq 0$, the 1 in the numerator can actually be dropped since $\int g = 0$. In particular, this remark shows that the x integral is rapidly decaying as $\lambda \to \infty$. Condition (2) admits the following reformulation:

(2′) For every $h \in C_0^\infty(\mathbb{R})$, the integral $\int d\sigma(\lambda) \int dx\, h(x) \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$ converges absolutely:

$$\int_{-\infty}^{\infty} d|\sigma|(\lambda) \left| \int_{-\infty}^{\infty} dx\, h(x) \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right| < \infty.$$

Moreover, there exists an odd, real valued function $\phi' \in L_{1,\text{loc}}(\mathbb{R})$, so that

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \int_{-\infty}^{\infty} dx\, h(x) \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} = -\int_{-\infty}^{\infty} h(x)\phi'(x)\, dx$$

for all $h \in C_0^\infty(\mathbb{R})$.

To prove that (2) and (2') are equivalent, observe that $g \in C_0^\infty(\mathbb{R})$, $\int g = 0$ precisely if $g = h'$ for some $h \in C_0^\infty(\mathbb{R})$, and integrate by parts.

As expected, we denote by GL the set of ρ's satisfying conditions (1) and (2) (or equivalently, (1) and (2′)).

Theorem 19.1. (a) *For every $\rho \in GL$, there exists a unique $V \in L_{1,\text{loc}}([0,\infty))$ so that ρ is a spectral measure of $-d^2/dx^2 + V(x)$.*

(b) *If ρ is a spectral measure of $-d^2/dx^2 + V(x)$, then $\rho \in GL$.*

Sketch of proof. (a) Proceed as in the proof of Theorem 17.1(a). The only place where an additional observation is needed is at the beginning of the argument. Here, we put, for $f \in C_0^\infty(\mathbb{R})$, $F(\lambda) = \int_{\mathbb{R}} f(x) \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\, dx$, and we want to show by a calculation that

$$\int_{\mathbb{R}} |F(\lambda)|^2\, d\sigma(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dt\, \overline{f(x)} f(t) (\phi(x-t) - \phi(x+t)). \tag{19.1}$$

This works well for odd f, and this is already the general case because we can decompose an arbitrary f into odd and even parts and the contributions containing even functions are zero on both sides of (19.1).

(b) As in the proof of Theorem 17.1(b), condition (1) is obvious from the defining property of spectral measures. If ρ is a spectral measure, then

$$\int_{-\infty}^{0} e^{L\sqrt{-\lambda}}\, d\rho(\lambda) < \infty \quad \forall L > 0, \qquad \int_{\mathbb{R}} \frac{d\rho(\lambda)}{1+\lambda^2} < \infty.$$

See [15, Section 6]; actually, this reference uses the term "spectral measure" in a slightly more restrictive sense, but the method of proof extends to our setting if combined with the Nevanlinna type parametrization of the spectral measures from [9].

The above estimates imply the absolute convergence of the integral from condition (2). As in the proof of Theorem 17.1(b), one shows that with ϕ being the ϕ function corresponding to the given potential V, the desired identity

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \int_{-\infty}^{\infty} dx\, g(x) \frac{\cos\sqrt{\lambda}x - 1}{\lambda} = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx$$

holds for functions g that are of the form $g = f_1 * f_2$ with odd functions $f_i \in C_0^\infty(\mathbb{R})$. Alternately, if we pass to formulation (2') by integrating by parts, this says that

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \int_{-\infty}^{\infty} dx\, h(x) \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} = -\int_{-\infty}^{\infty} h(x)\phi'(x)\, dx \tag{19.2}$$

for all $h = F_1 * f_2$ with F_1 even, f_2 odd, $F_1, f_2 \in C_0^\infty(\mathbb{R})$. (Of course, F_1 is just $F_1(x) = \int_{-\infty}^{x} f_1(t)\, dt$.) By the argument from the proof of Theorem 17.1(b), (19.2) now follows for all odd $h \in C_0^\infty(\mathbb{R})$. For even $h \in C_0^\infty(\mathbb{R})$, (19.2) is trivially satisfied: both sides are equal to zero. $\square$

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